VOLUME LII

The Mathematics Teacher

NUMBER FIVE

MAY 1959

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THE SECONDARY-SCHOOL CURRICULUM COMMITTEE

The official journal of
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Mathematics Teacher *is the official journal of The National Council of Teachers of Mathematics devoted to the interests of mathematics teachers in the Junior High Schools, Senior High Schools, Junior Colleges, and Teacher Education Colleges.*

All editorial correspondence, including books for review, should be addressed to the Editor

All other correspondence should be addressed to

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N. W., Washington 6, D. C.

Printed at Menasha, Wisconsin, U.S.A. Entered as second-class matter at the post office at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412, P. L. & R., authorised March 1, 1930. Printed in U.S.A.

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Silken slippers and hobnailed boots, BURTON W. JONES 322
The theory of braids, EMIL ARTIN 328
Current school mathematics curricula in the Soviet Union and other Communist countries, IZAAK WIRSZUP 334
Providing for the slow learner in the junior high school, MAX A. SOBEL 347
Angle trisection—an example of "undepartmentalized" mathematics, REV. BROTHER LEO, O.S.F. 354
One plus one equals toon, FRANKLIN S. MCFEELY 356
Trisecting an angle, C. CARL ROBUSTO 358
Mechanics of orbiting, M. V. LANDON 361

DEPARTMENTS

Historically speaking,—HOWARD EVES 365
New ideas for the classroom, DONOVAN A. JOHNSON 371
Points and viewpoints 376
Reviews and evaluations, RICHARD D. CRUMLEY 378
Tips for beginners, JOSEPH N. PAYNE *and* WILLIAM C. LOWRY 384
Have you read? 333, 360; *What's new?* 353, 370
Letters to the editors, 355, 383, 388

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Nineteenth Summer Meeting 377
The Secondary Mathematics Curriculum 389
Officers of the NCTM Affiliated Groups 418
Your professional dates 425

THE MATHEMATICS TEACHER is published monthly eight times a year, October through May. The individual subscription price of $5.00 ($1.50 to students) includes membership in the Council. For an additional $3.00 ($1.00 to students) the member may also receive *The Arithmetic Teacher*. Institutional subscription: $7.00 per year. Single copies: 85 cents each. Remittance should be made payable to *The National Council of Teachers of Mathematics*, 1201 Sixteenth Street, N. W., Washington 6, D. C. Add 25 cents for mailing to Canada, 50 cents for mailing to foreign countries.

Silken slippers and hobnailed boots

BURTON W. JONES, *University of Colorado, Boulder, Colorado.*
The present situation in mathematics education has no parallel in the history of the schools. There is reason to be hopeful about future developments.

MY TITLE IS TAKEN from a quotation attributed to Voltaire: "History is but the pattern of silken slippers descending the stairs to the thunder of hobnailed boots climbing upward from below." This is a saying to make one shudder. It is good for us that the thunder of boots is very loud these days, but we also are mindful of many rising movements in history that were cut down by internal and external forces. Napoleon was a leader of a rising power that made the world tremble—but he had his Waterloo. Hitler was a man of destiny—a spiritual as well as a temporal leader—but he fell too, and the Nazi dreams of glory met a flaming death. So the other side of the picture is shown by a familiar scene from *The Talisman* (Chapter 27). You remember that Richard the Lion-hearted was the guest of Saladin, and each was extolling the prowess of his nation. Richard with one mighty blow of his heavy sword cleft a steel mace. But Saladin asked that a silken pillow be brought, and with one deft stroke of his slender scimitar sliced it in two.

Arnold Toynbee points out that though one can see some of the degenerative forces that have led to decay of civilizations in the past at work in our own, there is no reason why we cannot, if we will, set in action forces of preservation. He has written:

> Civilizations, I believe, come to birth and proceed to grow by successfully responding to successive challenges. They break down and go to pieces if and when a challenge confronts them which they fail to meet. Not unnaturally there are challenges that present themselves in the histories of more than one civilization. And the peculiar interest of Graeco-Roman history for us lies in the fact that the Greek civilization broke down in the fifth century B.C. through failing to find a successful response to the very challenge which is confronting our own Western civilization in our own lifetime.[1]

Toynbee goes on to indicate that the destruction of the Graeco-Roman civilization was due to failure to replace an international anarchy with some kind of international law and order. Such international order can today come about in one of two ways: by one nation's conquering the world or by the co-operation of all nations.

The present efforts of this nation are toward making itself strong—strong in its friends and strong internally—so that, first, no nation can conquer the world and so that, second, the passage of time and the threat of total destruction may force us and the rest of the world into a system of international law. One phase of this struggle is improving the education of our youth. And there are many facets of education: not only science and mathematics, but social sciences, languages and literature, and that all-important education outside of the classroom. It is probably true that the final victory of mankind will not be won in the laboratories but in the field of human relations, though the former seems to be exerting marked effect on the latter. There are significant and vital new programs in foreign languages as well. But this article will not be con-

[1] Toynbee, Arnold, *Civilization on Trial* (New York: Oxford University Press, 1948), pp. 56 ff.

cerned with international law, politics, religion, or the like. It will concern itself with the realm of education and, in particular, science and mathematics. This choice is not dictated by relative importance but by the competence of the author. Here will be discussed some of what he considers to be the degenerative forces at work today in American education so that we may strive to avoid them and some of what he considers to be the regenerative forces so that we may cultivate them. We shall see that some forces belong in both categories.

The first to be mentioned is our habit of self-criticism. A few months ago, in a popular magazine, appeared a most intemperate criticism of U. S. education. One of the select few in the Russian school system was contrasted with a not-so-bright American school boy. A picture was shown of a Russian blackboard with symbols of inequalities, and the comment was made: "A hard math taught in few U. S. schools." The author of the article (who is there identified as someone who is "best known as a novelist") also points with horror to the fact that only 25 per cent of our secondary-school students take physics. This is about one-fourth of the total population of school age. In England, however, about one-fifth of the boys and girls go to grammar schools and public schools. Only in such schools would a course be offered comparable to our physics course in high school. In one grammar school, which the author knew in London, less than one-third of the students took physics—the others were so unwise as to prefer the language courses or the social sciences. Certainly in England the proportion of the population of school age taking physics is much less than in this country!

These are just two examples of the misinformation and bias of the article. It is too bad that it will be read all over the world as an authoritative pronouncement on the decadence of the U. S. system of secondary education.

In London there is a little shop (probably more than one) where one can get all sorts of Communist publications, but they are not usually on sale at the newsstands there because not enough people want to buy them, and very rarely does one see a copy of *The Daily Worker* being read in the Underground (i.e., subways) of London. A look at some of the literature quickly reveals the reason—it just is not interesting. What one first notices is the complete lack of self-criticism. Any person reading it would realize that it must be false. Even in Russia, news of some dissatisfaction with its system of education is allowed to escape. Certainly the British are critical of their educational system—at many social gatherings among parents this is a heated subject of conversation. But somehow one does not hear this subject outside Great Britain. The Dutch do not like their school system either, and they are quietly trying to correct it.

Too much of the criticism of our educational system is irresponsible and uninformed. Those constructively working to improve the system have a duty also to educate the decriers. This has been done in many instances, and critics converted to workers, with the result that many scientists and mathematicians are now taking an active part in the education of the young in their communities and are incidentally becoming educated themselves. We cannot survive without self-criticism, but let us not despair. In a way this attitude of self-criticism is one of the most lovable characteristics of the American people: we are masters of advertising everything but ourselves. It is for this reason that our foes (and friends) always underestimate us. Certainly we must not underestimate ourselves.

Second, the word "democracy" sometimes takes on strange meanings in people's minds and arguments. It is proper that the best physical specimens in the school be given special instruction in football by a highly paid staff and in college be given special scholarships. In the game of football, no one is surprised that only a

select few are allowed to carry the ball. A boy whose father can buy him a car is "entitled" to have it. Students of low intelligence should be given special treatment by specially trained teachers. That this is part of democracy no one will deny. But attempts to separate gifted students are often called "undemocratic," and fear is expressed that those not selected will wilt under the brand of inferiority. Fortunately these voices are being heard less and less in the land, and special programs and inducements for gifted and superior students are sprouting everywhere and growing like weeds. Even in these programs we must watch out for the word "acceleration," since depth and inspiration are rather what we need. Henry T. Heald, president of the Ford Foundation, at an address given at the Sesquicentennial Dinner of John Wiley and Sons, said these ringing words:

> I think no head is so full that one idea must depart before another can enter. I believe no idea is so pristine that its opposite is unthinkable. And it encourages me to know that the flames arising from all the books ever burned have not the candlepower of one spark from an ignited imagination.

This strange conception of democracy is also used by some teachers and teachers' organizations to imply that all teachers should be treated alike. In some school systems a few teachers are given special courses and released time to prepare for these courses. It is hard to convince some of their colleagues that what benefits a few in time can benefit the whole profession. They may even object to the National Science Foundation summer institutes because the institutes make further education easier for some than it was for them. One of the worst faults of the school system is that a teacher's salary is determined more by how long he has taught than by how good he is. People in industry, for instance, would be willing to support increased salaries for good teachers but not across-the-board increases, for they feel that some teachers are not worth what they are getting. If a teacher of ten years' experience finds that he cannot provide for a growing family on his salary, he cannot move to another school system with a higher salary scale since only six years of his teaching would be recognized in setting his salary. He has two choices: become an administrator or change his profession. Appointment to administrative positions should not be the only way to reward outstanding teachers, for administration requires special training and ability, and there is too much risk of losing good teachers and getting poor administrators. Many excellent mathematicians would be completely helpless as chairmen of departments. And many a research man has been spoiled by being made a dean.

The arguments against a merit system are well known. One is that administrators are not to be trusted. Are school administrators of a different breed from those in colleges or industry? Faced with the necessity of making a choice, they would undoubtedly consult the chairmen of departments as they do in colleges. The chairmen would probably object, but assisting in such decisions should be one of their duties. There *is* the difficulty of selecting the best teachers, and fine gradations could not be made, but any superintendent or any principal knows who his outstanding teachers are, and most of the fellow teachers would agree. Surely teachers are unselfish enough to approve of special recognition of merit among their colleagues, and the disappointment of a few who consider themselves not properly recognized under a merit system must be balanced against the present feeling of a good teacher who sees a lazy colleague getting more salary just because he has been there longer.

By and large, special recognition of a deserving few should raise the morale of all, improve instruction, and add to the dignity of the teaching profession. There could be special "chairs" or titles for outstanding teachers. (In at least one private school there is such an arrangement.) It is not to be claimed that the system in col-

leges and universities in this respect is perfect, but an ambitious and capable college teacher can, especially in these times, better his position by moving elsewhere, and usually in collegiate institutions promotion is rapid for the capable.

In one locality a plan has recently been proposed which is in the nature of a compromise between the straight merit system and the automatic-increase system. Under this plan, a teacher who wishes to do more than the minimum during the school year applies to the school board or is approached by them. He may chair a committee, work on materials for changes in curricula, organize and sponsor a science club. The teacher would be paid for such extra work. Similarly, in the summer a teacher would receive extra salary for improving his knowledge of subject matter, attending an institute or conference, working on curriculum changes, conducting surveys, etc. *If we are to attract ambitious young men to secondary-school teaching, it is imperative that we give monetary recognition to merit.*

Third, looking from the outside on the interplay between administrators and teachers in the public school system, one may see some things that are disturbing and others which are most exciting. Some administrators are quick to jump on the bandwagon of a popular idea—it gives them something to write and talk about. This in itself is not bad, but oftentimes the administrators display their ignorance of what is already going on and show more interest in having something on the books than in taking time and effort to do it well. An administrator is merely succumbing to hysteria when he decrees that calculus shall be taught in his school without first ascertaining whether he has students prepared to take it or teachers to teach it.

On the other hand, teachers need at times to be poked out of their ruts. (This is true in colleges just as much as in high schools.) Often the teachers, having no program of their own to suggest, resist, as only they can, the efforts of their superiors. But more often teachers take the initiative, or jump at the opportunity that the sudden interest of their administrators presents. One of the virtues of our educational system is the relative autonomy of the schools in each locality. Changes can be wrought by teachers and administrators working together in school systems here and there without being made on a national scale and thereafter fixed forever. This flexibility of our school system is its greatest strength, and it is worth the price we pay in lack of uniformity. One cannot in a country of this size have both.

Finally, there is the present ferment in curriculum changes and course materials in science and mathematics. Though there may be degenerative tendencies evident, certainly the reactions are vigorous and varied, and it is very wholesome that they are. This freedom of reaction is certainly one outstanding characteristic of the "democratic process." Though important developments are occurring in biology, physics, and chemistry, this article will be largely confined to the subject of mathematics.

Here consider the various reactions to these new proposals. First, there are those who are downright hostile to *any* change. Their reasons are somewhat vague, even to themselves. Some are of the "practical" school—mathematics for a high-school student is to enable him to figure income tax, how much he is really paying in interest to a finance company, or how he can measure the height of a tree without climbing it. And there are those who feel that the only good system is the continental system of rigorous training for the few, following a pattern laid down one hundred years ago, and that any variations from it constitute a watering-down process. Even though they realize that mathematics is growing at a surprising rate and that the relative importance of different parts of the subject is changing, they seem to feel that there should be no change in what is taught except that it should be taught more thoroughly.

Second, there are those who feel that subjects which have been traditionally taught in college or graduate school are too difficult for the youthful mind. Of course, they are right if the subjects are dealt with as they would be in graduate school. But certain ideas of sets, for example, are much simpler than the concept of number to those unacquainted with both. Before a child can count, he can distinguish a set of black blocks from a set of white blocks, tell to which set a given block belongs, and determine whether there are more blocks in one set than in the other. We have what might be called the relativity principle of teaching: what a teacher finds simple he thinks his students should also find simple, and what a teacher finds difficult he thinks his students should also find difficult. But if the teacher were to try to teach a subject new to him, suddenly he would find his students becoming brighter. He might even have trouble keeping up with them, for they would have two big advantages: the flexibility of young minds and fewer preconceived notions. It is these two factors which lie behind the fact that a scientist's greatest discoveries are made while he is young.

We continually underestimate the ability of youngsters to form abstractions. There is much shaking of heads at the well-publicized experiment in the teaching of mathematics at the University of Illinois. It is the author's opinion that the experimenters have been the victims of bad publicity that emphasizes the spectacular and strange and omits the solid work that is being done. (Recommended is their own description of their work in *The School Review*.[2]) Certainly it is a thrilling experiment, made in bold recognition of the obvious fact that as mathematics grows, we shall learn a smaller and smaller proportion of the subject unless some means can be devised to arrive more quickly at important things. To retrace the paths our forefathers trod is not appropriate to a space age.

Last there are those who are afraid. They say: "Here are these new things which I do not understand—not even the language carries meaning to me. Am I suddenly to become incompetent after all these years of teaching?" Here reassurance is in order. Actually the changes proposed in mathematics are of three kinds, and in the author's opinion they are listed as follows in *decreasing order* of importance: (1) Fresh points of view toward traditional materials. (2) Changes of emphasis in the traditional curriculum. For instance, there should be less stress on the computational aspects of logarithms and more on the idea of a logarithm as a function which replaces multiplication with addition; inequalities and their graphs are of increasing importance. (3) Subjects new to the secondary-school curriculum, such as probability and statistical inference. In the first place, no one in his right senses would advocate that such changes should be introduced before a teacher is qualified to teach them. The programs for change in secondary schools and colleges are looking far ahead and can come only gradually as teachers learn the new points of view and by process of experiment find which ones work and which ones do not.

Curriculum reforms have been advocated before. But this movement is different for a number of reasons. First of all, those advocating changes are not content to issue fiats from above: college people and secondary-school teachers are sitting down together to draw up specific experimental materials for use in the secondary schools. Well known is the physics group at M.I.T. A similar group in mathematics began at Yale University this past summer. Second, the point of view is not, "This is it," but rather, "Something like this might be it—try it out and see how it works." Various summer institutes supported by industry and the massive program of the National Science Foundation of summer and academic-year institutes

[2] *The School Review*, LXV (Winter, 1957).

all have the same object: to encourage teachers to renew their study of subject matter from a fresh point of view and, incidentally, to inspire the colleges to develop courses appropriate for teachers and, in general, to reform their own curricula.

Evidence is that these institutes will work hand in glove with informal and formal writing groups in trying out these new points of view and new materials. The Commission on Mathematics of the College Entrance Examination Board, as you probably know, is not advocating sudden changes. The Commission was formed because it was felt that the College Entrance Examinations were, by their unchangeableness, exerting a stultifying influence on the mathematics curriculum, and that they should instead encourage new points of view. The Commission has consulted groups of persons all over the country so that its recommendations may be revised in the light of these discussions. The Commission fully realizes that what it proposes must be tested in the crucible of experience and that there is no one answer to the question of what should be taught. The group which began its work at Yale (the School Mathematics Study Group) has a similar point of view toward the materials which it is developing.

Of course, some experiments go off half-cocked under stress of a crash program. But this cultivation of the grass roots is one of the most thrilling events of modern times. And it is being done with gentleness —no one is forced. Beyond prescribing the size of stipends and relative size of operating expense, the National Science Foundation does not interfere in the detailed operation of its institutes. The object of the Foundation is to assist institutions to develop their own programs, but it does not dictate what those programs shall be, except that they shall be primarily concerned with content.

Perhaps we have stressed too much the hopeful signs of these times. But these remarks have been based on the belief that the pull of Heaven is stronger than the fear of Hell. It is certain that most of the scientists from colleges, universities, and industry who are now working so hard will in time become tired and return to their primary interests. Any lasting effect of this ferment depends on what happens when "the tumult and the shouting die" and "the Captains and the Kings depart." The fate of this attempt at regeneration is in the hands of the teachers in elementary and secondary schools. It is the teachers who will answer the question: Is this to be a sawdust trail forgotten in the slumbers of the morning after, or a mighty surge of many minds that sets the world on fire?

The surge of publicity about Soviet schools has produced more false impressions and foolish conclusions than almost any other element in current discussions of education.—*James B. Conant,* The American High School Today.

*The theory of braids**

EMIL ARTIN, *Princeton University, Princeton, New Jersey.*
Professor Artin's famous lecture is reprinted because the Editor feels secondary teachers will be interested in a paper which is easily understood and uses some very important mathematical concepts.

THE THEORY OF braids shows the interplay of two disciplines of pure mathematics—topology, used in the definition of braids, and the theory of groups, used in their treatment.

The fundamentals of the theory can be understood without too much technical knowledge. It originated from a much older problem in pure mathematics—the classification of knots. Much progress has been achieved in this field; but all the progress seems only to emphasize the extreme difficulty of the problem. Today we are still very far from a complete solution. In view of this fact it is advisable to study objects that are in some fashion similar to knots, yet simple enough so as to make a complete classification possible. Braids are such objects.

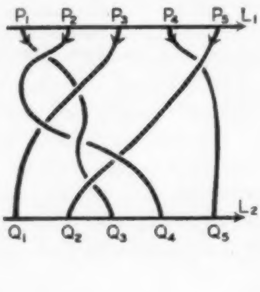
In order to develop the theory of braids we first explain what we call a *weaving pattern* of order n (n being an ordinary integral number which is taken to be 5 in Fig. 1).

Let L_1 and L_2 be two parallel straight lines in space with given orientation in the same sense (indicated by arrows). If P is a point on L_1, Q a point on L_2, we shall sometimes join P and Q by a curve c. In our drawings we can only indicate the projection of c onto the plane containing L_1 and L_2, since c itself may be a winding curve in space.

The curves c that we shall use will be restricted in their nature by the following condition. If R is a point on the projection of c that moves from P to Q, then its distance from the line L_1 shall always increase. (Therefore a curve moving down a little, then up, and finally down again would be ruled out.) In order to have at our disposal a short name for such curves, let us call them normal curves. We orient them (by arrows) in the sense from P to Q.

Select n points on L_1. Moving along L_1 in the direction indicated by the arrow we shall call the first of the given n points P_1, the next P_2, and the last P_n. In the same way denote by $Q_1, Q_2, \ldots, Q_n$, n points on the line L_2. Now we connect each point P_i with one of the points Q_j by a normal curve c_i (c_1 begins at P_1 and ends at some Q_j, which may or may not be Q_1). We only observe the following condition: no two of the curves c_i intersect in space. Conse-

Figure 1



* Reprinted with the permission of *The American Scientist* and the author. This paper appeared in *The American Scientist*, Volume 38, No. 1, January, 1950, pages 112 to 119.

quently no two of the curves c_i end at the same point Q_j.

If we want to indicate this in a drawing, we have to overcome the difficulty that, although the curves do not meet in space, their projections may cross over each other. To indicate that at a certain crossing the curve c_i is below another one, we interrupt its projection slightly (this is the well-known way to indicate such occurrences in technical drawings).

The whole system of straight lines and curves shall be called a weaving pattern.

In order to explain the notion of a braid we start with a given weaving pattern and think of the lines L_1 and L_2 as being made of rigid material, whereas the curves c_i are considered as arbitrarily stretchable, contractible, and flexible. The points P_i and Q_j may also move on their lines provided their ordering is always preserved.

We subject the whole weaving pattern to an arbitrary deformation in space restricted by the following conditions:

1. L_1 and L_2 stay parallel during the deformation (but otherwise they can be moved freely in space; their distance may change).
2. No two of the curves c_i intersect each other during the deformation (this means that the material is "impenetrable").
3. The curves stay normal during the deformation (but otherwise they may be stretched or contracted as the situation demands).

After such a deformation we obtain a weaving pattern that may look quite different from the one we started with. A quite tame-looking pattern may indeed (after the deformation) become hopelessly entangled.

By a braid we mean a weaving pattern together with the permission to deform it according to the previous rules. If we present a weaving pattern, it describes a braid. But infinitely many patterns will describe the same braid, namely, all those that can be obtained from the given

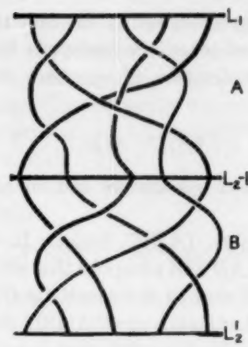


Figure 2

one by a deformation. The order n of the pattern shall be called the *order of the braid*.

We now have the following fundamental problem. Given two weaving patterns, is it possible to decide whether or not they describe the same braid? In other words, is it possible to decide whether or not a pattern can be deformed into a given other one?

Up to now we have considered braids of all orders n. From now on we assume n to be an arbitrary but fixed integer and restrict ourselves, without saying it explicitly, to braids of that order n.

Let now A and B be two braids. We first explain what we mean by the product AB of A and B. We select definite patterns for A and B. Call L_1, L_2, P_i, Q_i, c_i the lines, points, and curves respectively of A, and $L'_1, L'_2, P'_i, Q'_i, c'_i$ those of B.

We deform B until the plane through L'_1 and L'_2 coincides with the plane through L_1 and L_2, and until the line L_2 coincides (including orientation) with the line L'_1, being careful to have L_1 and L'_2 on different sides of L_2. Finally we deform B until the points Q_i coincide with the points P'_i. This being achieved, we erase the line L_2, obtaining a new composed weaving pattern which shall stand as pattern for the braid AB.

Intuitively speaking, this means: AB is obtained by tying the beginning of B to the end of A. Figure 2 explains the process. The reason for calling the result of this

process a product lies in the fact that the process has some similarity to the ordinary multiplication of numbers. We first show:

$$(AB)C = A(BC),$$

the so-called associative law of multiplication.

What does $(AB)C$ mean? It means: form first AB and compose this with C. So tie B to A and to the result tie C. What, on the other hand, does $A(BC)$ mean? It asks us first to form BC, that is to tie C to B. The result shall be tied to A. Obviously we obtain the same pattern as $(AB)C$.

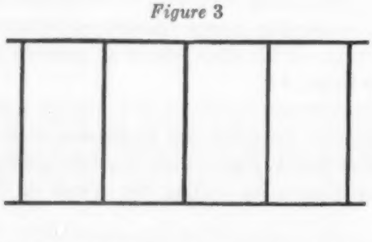
But this similarity does not go too far. For instance, the law $AB = BA$ is false in general. Very simple examples already show this. It may hold only accidentally for very special braids. In computations one must, therefore, be careful about the order of terms in a product.

Let us denote by I the braid indicated in Figure 3. In its pattern the curves c_i are simply straight lines joining P_i and Q_i without crossings. If we tie I to any braid A, it is almost immediately seen that the resulting braid AI can be changed back to A; indeed the line L_2 is simply replaced by a somewhat lower line. Therefore $AI = A$ for any braid A; similarly we see $IA = A$ for any A.

Our braid I has therefore a strong resemblance to the number 1 (since $1 \cdot a = a \cdot 1$ for any number a). This explains the choice of the name I (roman one).

What does the equation $A = I$ mean? If A is originally given by some complicated pattern, then $A = I$ means that by some

Figure 3



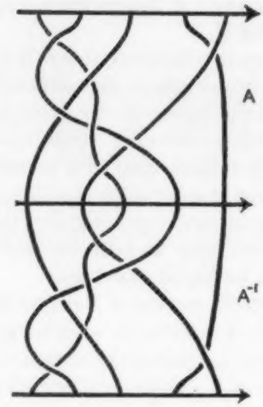


Figure 4

deformation this pattern can be changed into the pattern of Figure 3. We may say intuitively: $A = I$ means that A can be combed.

Figure 4 shows the braid A of Figure 1, and tied to it its exact reflexion on the line L_2 which we call A^{-1}. The reader can convince himself that the combined braid AA^{-1} can be disentangled if he starts removing crossings from the middle outward. In the same way he can see that $A^{-1}A$ can be combed.

There exists therefore to any braid A another braid A^{-1} (its reflexion) such that

$$AA^{-1} = A^{-1}A = I.$$

The symbol A^{-1} is chosen because of an analogy with elementary algebra where a^{-1} stands for the number $1/a$ so that $aa^{-1} = a^{-1}a = 1$ for any non-zero number a.

Reviewing we may say: the braids form a system of objects in which a multiplication is defined. Three properties hold for this multiplication:

(1) The associative law $(AB)C = A(BC)$ is satisfied.

(2) There is a braid called I such that $AI = IA = A$ holds for any braid A.

(3) To any braid A another braid A^{-1} can be found such that $AA^{-1} = A^{-1}A = I$.

If in these three statements we were to replace the word "braid" by the phrase

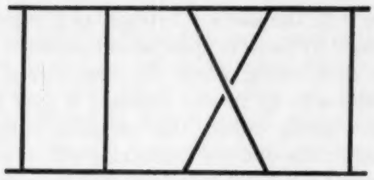


Figure 5

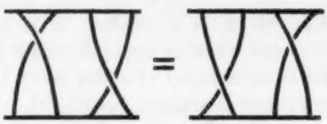


Figure 6

"object of the system," we should obtain the exact definition of what in higher algebra one calls a "group." A group is simply a system of arbitrary objects, together with some kind of multiplication such that our three properties hold. We may say therefore: the system of all braids of order n is a group.

The theory of groups has been developed extensively, and its methods may be applied to our problem. Let us look at the special braid indicated in Figure 5. Here the curve c_i goes once over the curve c_{i+1}, whereas all other curves are straight lines connecting P_j and Q_j. We shall call this braid σ_i and obtain in this fashion $n-1$ braids $\sigma_1, \sigma_2, \cdots, \sigma_{n-1}$ (σ_n does not exist since it would involve an $n+1$-st curve). The braid where c_i goes *under* c_{i+1} needs no new name. It is the reflexion of σ_i and may therefore be denoted by σ_i^{-1}.

Consider now the pattern of any braid A, for example, the braid in Figure 1. In its projection two crossings may occur at exactly the same height. But it is evident that a slight deformation of braid A will produce a pattern where this does not happen.

We cut up our pattern into small horizontal sections, such that only one crossing occurs in each section. Our braid A is obtained from all these sections by tying them together again. Each of these sections is obviously either a braid σ_i or a braid σ_i^{-1} depending on the nature of the crossings. Consequently we can express A as the product of terms each of which is either a σ_i or a σ_i^{-1}.

The braid in Figure 1, for example, is given by:

$$A = \sigma_1^{-1}\sigma_4^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_3\sigma_2^{-1}.$$

If every element in a group can be expressed as product of some elements σ_i and their inverses, we say that the σ_i are generators of the group. We may therefore state: the $n-1$ elements σ_i are generators of the braid group.

We are now in a position to describe any weaving pattern. As an example, let us look at the braids in a girl's hair. A close look reveals that such a braid can be described by:

$$A = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1} \cdot \cdot \cdot \sigma_1\sigma_2^{-1} = (\sigma_1\sigma_2^{-1})^k,$$

where k is the number of times the elementary weaving pattern is repeated.

Figure 6 shows the equality $\sigma_1\sigma_3 = \sigma_3\sigma_1$. A similar figure would show $\sigma_i\sigma_j = \sigma_j\sigma_i$ if j is $i+2$ or more. That $\sigma_1\sigma_2$ is different from $\sigma_2\sigma_1$ can be seen by a simple sketch; in $\sigma_1\sigma_2$ the curve c_1 runs from P_1 to Q_3, whereas in $\sigma_2\sigma_1$ it runs from P_1 to Q_2.

But $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. Figure 7 shows it for $i=1$; the reader readily deforms the two patterns into each other.

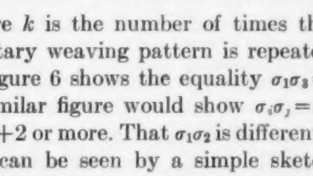
The formulas:

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$, if j is at least $i+2$

(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

have the following significance:

Figure 7



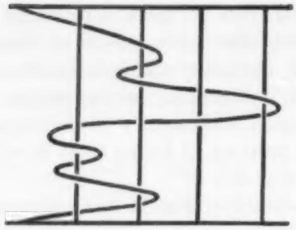


Figure 8

Suppose two braids A and B given by patterns. Each pattern may be used to express A and B, respectively, as a product of terms σ_i or σ_i^{-1}.

If $A=B$, it must in some fashion be possible to change from the expression A to the expression B. It can be shown that this can always be done by a repeated use of either formulas (1) or (2), or of simple algebraic consequences of these formulas. It is this fact one refers to if one says: the braid group has the defining relations (1) and (2). The proof is too long to be reproduced here.

We proceed now to our fundamental problem. Let us first consider a braid A in which the curves c_i connect P_i with Q_i (the Q_i with exactly the same subscript).

Suppose we remove the curve c_1. A certain braid A_1 of order $n-1$ remains. Now we reinsert a curve d_1 between P_1 and Q_1 that is not entangled at all with the other strings (this means that its projection exhibits no crossings at all). This new braid of order n we call B.

Denote now the braid AB^{-1} by C. This braid C has a peculiar property. If the first string of C is removed, then the braid that remains from the A-part of C is A_1, and A_1^{-1} is the part that remains from B^{-1}. (According to our construction, A and B differ only by their first strings.) Therefore removing the first string from C leaves $A_1 A_1^{-1}=I$—a braid that can be combed. To be sure, C itself cannot necessarily be combed until the first string has been removed.

Suppose now that this combing operation with the last $n-1$ strings of C is performed by force in spite of the presence of the first string. Since the first string is stretchable up to any amount, it may be taken along during this combing operation. At the end the first string will be entangled in a terrible fashion, but the result will look somewhat like Figure 8. A pattern of this type is called 1-pure.

Now $AB^{-1}=C$; $AB^{-1}B=CB$; therefore $A=CB$. So A is a product of a 1-pure braid C and another braid B which is obtained from a braid of order $n-1$ by inserting a first string not meeting the others in a projection. The second string of B can be treated in the same way, and so on.

The final result is:

$$A=C_1C_2 \cdots C_{n-1},$$

where C_i is a braid of the following kind: all strings but the i-th are vertical straight lines, and the i-th is only involved with strings of a higher number. Of course this means that for every braid A a pattern of this special kind can be found.

The solution of our fundamental problem consists in the assertion that a pattern of this type describes the braid uniquely, i.e., that in order to test whether $A=B$ for two braids whose curves c_i connect P_i with Q_i one has only to bring A and B into this form and to see whether exactly the same pattern results. The proof for this fact is very involved and cannot be included here. Nor shall we describe the translation of our geometric procedure into group theoretical language.

It is clear that this procedure contains the solution of the full problem to decide whether $A=B$ for any two braids A and B given by weaving patterns. First $A=B$ means the same as $AB^{-1}=I$. The braid I connects P_i with Q_i. Should AB^{-1} not do this, then certainly A is not equal to B. In case AB^{-1} connects each P_i with Q_i, the previous method makes it possible to decide whether $AB^{-1}=I$ or not.

Finally let us mention an unsolved problem of the theory of braids. If we wind a braid once around an axis, close it

by identifying P_i and Q_i, and remove the lines L_1 and L_2, we obtain what we call a closed braid. Again we allow all those deformations in the course of which the curves do not cross the axis, nor each other.

The problem of classification of closed braids, at least, can be translated into a group theoretical problem. Let A and B be two open braids. The corresponding closed braids are equal if, and only if, an open braid X can be found such that

$$B = XAX^{-1}.$$

A solution to this problem has not yet been found. Since in some ways closed braids resemble knots, such a solution could be applied to the problem of knots. It would also have many applications in pure mathematics.

Have you read?

Harrison, Melvin. "La Puente Challenges the Superior Student," *California Journal of Secondary Education*, February 1959, pp. 68–71.

You have read a lot about the superior student, but it seems this article may have something to offer. In this school the students apply for admission. Of the 105 who applied 75 were accepted. They must be in the superior group in three areas of study to be retained.They take great pride in their group, and effort must be made by school officials to arrange for association with others. In mathematics in grade ten they take plane geometry, in grade eleven, trigonometry, solid geometry and surveying, and in grade twelve they take analytic geometry and calculus. This is not a different program, but it covers much material in these areas. All students have permission to withdraw at any time, but so far none have. They also accept the A, B, C range of grades even though most of the students have been in the straight A class before entering this group. This program also helps assure the student success in college. You will want to read this for ideas.—Philip Peak, *Indiana University, Bloomington, Indiana.*

Satvetz, Aaron. "Education and Scientific Knowledge," *Journal of General Education*, October 1958, pp. 192–96.

This article presents a healthy point of view about science education. Although it is not strictly about mathematics, I think you will appreciate reading it. The author separates training and education by saying, "Man is educated for life and trained for a job." The dilemma comes when one considers the place of education in training. Some questions might be raised, such as, "Should we train more scientists?" "Does training impede education?" "Have you thought about the purpose of education in science as reaching toward a goal that recedes as your knowledge of science increases?" The author says scientific knowledge is not the summation of separate inquiries; rather it is an amalgamation or transformation. It would seem that in this great social activity, we need to teach exemplifying the dynamic aspect of scientific inquiry. The object of education is to develop one's ability to form judgment from the inside. Read the article and see if you can apply it to your mathematics classes.—Philip Peak, *Indiana University, Bloomington, Indiana.*

And I cherish more than anything else the Analogies, my most trustworthy masters. They know all the secrets of Nature, and they ought to be least neglected in Geometry.—*Kepler*

Current school mathematics curricula in the Soviet Union and other Communist countries[1]

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Russians follow developments in American education with interest. Most certainly we should acquaint ourselves with the basic Russian curricular pattern.

THE SOVIET UNION

IN THE SOVIET UNION, the school for the full elementary and secondary general education[2] hitherto included ten grades,[3] and is called a "ten-year school" (or "complete middle school"). A school which has only the first four grades is called an "elementary school"; one having the first seven grades is called a "seven-year school" (or "incomplete middle school"). Since 1943, children must be seven years old before they may enter the first grade. At present, the seven-year school is compulsory throughout the Soviet Union, and ten-year schooling has been made compulsory in all urban areas in the last few years. In 1956, the Twentieth Congress of the Communist Party decided that compulsory ten-year schooling should be achieved essentially throughout the entire Union by 1960.

N. S. Khrushchev's proposals, "On the strengthening of ties between school and life and the further development of the public education system," which were adopted at the end of December 1958 as the law of the land by the Supreme Soviet, have introduced most radical changes in the whole educational system and have now established compulsory eight-year schooling.

The academic year of the general-education schools consisted of 33 six-day weeks. In the first three grades there were 24 class hours per week; in the last six grades the average was 34 class hours per week (32–36), each class meeting lasting 45 minutes.

There are both academic and practical emphases in the general-education schools. For the brighter students, these schools are the paths to the institutions of higher learning; for other students, the schools are preparation for practical life. Russian leadership is acutely aware that the level of professional manpower—from the common worker to the highly skilled specialist—is directly dependent upon the quality of the education received. Thus the educa-

[1] This paper is part of a Survey of Recent East European Literature in Intermediate Mathematics. The Survey is being conducted by the College Mathematics Staff of the University of Chicago and is sponsored by the National Science Foundation.

[2] Professional and semiprofessional schools lie outside of the scope of this paper.

[3] Except in the three Baltic republics and the Georgian republic. Here an additional year of preparation extends the usual scheme to an eleven-year system.

tional system is designed to serve the State's aims, to raise the country to a higher cultural level, and to promote the State's economic and technological progress, its military strength, and its political influence.

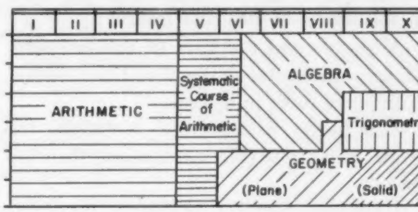
As might be expected from this orientation, the greatest emphasis in the school curriculum is placed upon the sciences. Mathematics, in particular, has been elevated to a position of supreme importance. Without exception, all students have fixed requirements in mathematics. In the regular ten-year school *20.5 per cent of school time* (1980 class hours) is devoted to this subject. In each grade of the general-education schools, one class hour per day is assigned to mathematics.

Months in advance of each academic year the Ministry of Education of the RSFSR (Russian Soviet Federated Socialist Republic) publishes a syllabus for the curriculum of the elementary school, and separate syllabi for each subject in the curricula of grades V–X. Shortly thereafter the corresponding ministries of the other Soviet republics do the same.

The mathematics syllabus includes an enumeration of the topics to be taught in each grade and a description of their content specifying the number of hours to be spent on classwork and homework. About half of this syllabus consists of explanatory notes setting out the aims and character of secondary mathematics in general and of each topic in particular. General and special methods of dealing with the material are also indicated, great emphasis being laid on the connection between mathematics and practical problems. These syllabi are effectively governmental decrees; their requirements must be satisfied within the prescribed time.

The mathematics program which every student in the ten-year school had to complete is shown at the bottom of the page.

This scheme is illustrated in the following diagram:



The 1957–58 syllabus for elementary schools of the RSFSR allotted six class hours per week to the study of arithmetic and elements of visual geometry in each of grades I–IV. This amounts to 198 class hours per year, or 792 class hours over four years, and represents *24.5 per cent* of the total 3234 class hours spent in the first four years.

	Grades	Total number of class hours	Total number of hours for homework
Arithmetic (with elements of visual geometry)	I–IV	792	—
Systematic course of arithmetic	V–VI	264	111
Algebra	VI–X	446	218
Geometry[4]	VI–X	346	164
Trigonometry	IX–X	132	63
Total School Mathematics		1980	556

[4] Plane geometry in grades VI–IX with a total number of 236 class hours and 111 hours for homework; solid geometry in grades IX–X with a total number of 110 class hours and 53 hours for homework.

Mathematics syllabus for the elementary school[5]

Grade I (age 7–8)

Counting up to 10; familiarity with numbers 1 through 10. Addition and subtraction within the limits of 10. (66 class hours)

Reading and writing numbers up to 20. Addition table. Increasing and decreasing a number by a few units. Multiplication within the limits of 20. Dividing numbers up to 20 into equal parts. (100 hours)

Reading and writing numbers up to 100. Addition and subtraction of tens within the limits of 100. Multiplication and division of tens by a one-digit number within the limits of 100. (22 hours)

Measures and exercises in measurement. Meter, centimeter. Kilogram. Liter. Week, number of days in a week.

Familiarity with square, rectangle, triangle, and circle (recognition and discrimination).

Problems. Solving problems involving one operation: to find a sum, a difference, to increase and decrease a number by a few units, to find a product (when a summand number is repeated several times), to divide a number into equal parts. Solving problems involving two operations.

Review of the material covered. (10 hours)

Grade II (age 8–9)

Review of the material of grade I. (12 hours)

Addition and subtraction within 100. Comparison of numbers by subtraction. (40 hours)

Multiplication and division within 100: familiarity with division of measurement type; multiplication table and division with the aid of a table. (72 hours)

Increasing a number "so many" times; decreasing a number "so many" times; finding a fraction of a number; comparison of numbers by division. (15 hours)

Multiplication and division within 100 without the aid of tables. (25 hours)

Reading and writing numbers up to 1000. The four arithmetic operations on whole hundreds within 1000, with the use of oral methods of computation. (16 hours)

Measures and exercises in measurement. Measures of length: kilometer, meter, centimeter. Measures of weight: kilogram, gram. Measures of time: year, month, day, hour, minute. (6 hours)

Straight line. Straight line segment and its measurement.

Problems. Solving simple problems: on comparison of numbers by subtraction; division of measurement type; on increasing and decreasing a number some number of times; on finding a fraction of a number; on comparison of numbers by division.

Solving composite problems involving two or three operations.

Review of the material covered. (12 hours)

Grade III (age 9–10)

Review of the material of grade II. (12 hours)

The four operations on whole tens and hundreds within the limits of 1000 with the use of oral methods of computation.

Written computations within 1000: addition and subtraction of three-digit numbers; multiplication of two- and three-digit numbers by one-digit numbers; table division within 100 with remainders; division of a three-digit number by a one-digit number. (44 hours)

Reading and writing numbers up to one million. Addition and subtraction; multiplication and division of many-digit numbers by one-, two-, or three-digit numbers.

Addition and subtraction on the abacus.

Identifying the components of arithmetic operations. Checking the results of operations. Order of performing arithmetic operations and parentheses (simple cases). (98 hours)

Measures and exercises in measurement. Table of measures of length: kilometer,

[5] Programmy nachalnoi shkoly na 1957–58 uchebnyi god, 1957.

meter, decimeter, centimeter, millimeter. Table of measures of weight: ton, centner, kilogram, gram. Table of measures of time: century, year, month, day, hour, minute, second. (5 hours)

Geometric material. Measurement of segments. Simple measurements on location: extending and measuring straight lines. Exercises in visual estimation of length.

Rectangles and squares; their sides and angles. Drawing a right angle, a square, and a quadrilateral with the use of a ruler and protractor. (8 hours)

Oral computations. Fluent calculations within 100 and by tens and hundreds within 1000. Using methods for rounding-off numbers, and the commutative property of addition and multiplication in oral computations.

Problems. Solving simple arithmetic problems and compound ones involving 2–5 operations (in close connection with the study of arithmetic operations).

Solving problems involving the simple rule of three; on proportional division; on finding an unknown from two differences; on two opposite motions. (19 hours[6])

Review of the material covered. (12 hours)

Grade IV (age 10–11)

Review of the material of grade III. (12 hours)

Reading and writing numbers including millions and billions. Categories and classes. Addition and subtraction of large numbers; commutative property of addition; relationship between the components of addition and subtraction; checking addition and subtraction.

Addition and subtraction on the abacus.

Multiplication and division of large numbers: commutative property of multiplication; relationship between the components of multiplication and division; checking multiplication and division; order of performing arithmetic operations (review). (44 hours)

Composite denominate numbers. Simple and composite denominate numbers. Breaking down and transforming denominate numbers in the metric system of measures. The four arithmetic operations on composite denominate numbers with metric measures. Problems involving all operations on composite denominate numbers. (26 hours)

Geometric material. Introduction to area. Units of measurement of area. Measuring and computing the area of a rectangle and square. Table of quadratic measures. Are and hectare. Solving problems involving computations of areas. Construction on location of a right angle, square, and rectangle. (14 hours)

Cubic measures. Familiarity with a cube: faces, edges and vertices of a cube. Cube as unit of measurement of volume. Measuring and computing the volume of right-angular bodies (boxes, rooms). Table of cubic measures. Solving problems involving computation of volume. (14 hours)

Measures of time. Table of measures of time (review); breaking down and transforming measures. The four operations on composite denominate numbers with measures of time (simple cases).

Problems involving the computation of time: within a day, a year, a century (the last in terms of whole years). (26 hours)

Simple fractions: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{5}$, $\frac{1}{10}$. Formation of parts. Numerator and denominator of a fraction. Transformation of fractions. Addition and subtraction of single and multiple parts. Solving problems involving the finding of several parts of a number. (18 hours)

Oral computations. Fluent calculations within 100, and within 1000 by tens and hundreds. Simple cases of successive multiplications and divisions (by 2, 4, 8, etc.). Short multiplication by 5, 50, 25.

Problems. Solving compound arithmetic problems involving 2–6 operations. Problems on finding the arithmetic mean. Problems solved by ratios. Finding two

[6] These are the hours allotted for the solving of typical problems only.

numbers from their sum and ratio. (15 hours)

Review of the material covered. (29 hours)

Following is a *summary of the topics of instruction excerpted from the mathematics syllabus of Russian secondary schools for the academic year 1957–58*[7] (the syllabus itself is too long for reprint here):

[7] Programmy srednei shkoly na 1957–58 uchebnyi god. Matematika. 1957.

Grade V (age 11–12)

Arithmetic (6 class hours per week, 198 class hours total)

	Class hours	Homework
1. Whole numbers	20	8
2. Divisibility of numbers	20	8
3. Ordinary fractions	90	36
4. Decimal fractions	50	20
5. Practical exercises	6	—
6. Review	12	6
Total	198	78

Grade VI (age 12–13)

(6 class hours per week, 198 class hours total)

Arithmetic (4 class hours per week in the first semester)

	Class hours	Homework
1. Percentage	20	10
2. Proportions. Direct and inverse proportionality	32	16
3. Review	14	7
Total	66	33

Algebra (4 class hours per week in the second semester)

	Class hours	Homework
1. Algebraic expressions. Equations	16	8
2. Positive and negative numbers	20	10
3. Operations on algebraic expressions	30	15
Total	66	33

Geometry (2 class hours per week)

	Class hours	Homework
1. Fundamental concepts	14	7
2. Parallelism	16	8
3. Triangles	32	16
4. Practical exercises	4	—
Total	66	31

GRADE VII (AGE 13–14)

(6 class hours per week, 198 class hours total)
Algebra (4 class hours per week)

	Class hours	Homework
1. Factoring polynomials	36	18
2. Algebraic fractions	24	12
3. Linear equations with one unknown	34	17
4. Equation with two unknowns. Systems of equations	28	14
5. Review	10	5
Total	132	66

Geometry (2 class hours per week)

	Class hours	Homework
1. Quadrilaterals	26	13
2. Circles	34	17
3. Practical exercises	6	—
Total	66	30

GRADE VIII (AGE 14–15)

(6 class hours per week, 198 class hours total)
Algebra (4 class hours per week in the first semester, 3 class hours per week in the second semester)

	Class hours	Homework
1. Powers and roots	44	22
2. Quadratic equations and equations reducible to them	42	21
3. Functions and graphs	12	6
4. System of second-degree equations	18	9
Total	116	58

Geometry (2 class hours per week in the first semester, 3 class hours per week in the second semester)

	Class hours	Homework
1. Relation and proportionality of segments	10	5
2. Homothety and similarity	18	9
3. Metric relationships in a triangle and a circle	36	18
4. Measuring areas of polygons	14	7
5. Practical exercises	4	—
Total	82	39

GRADE IX (AGE 15–16)

(6 class hours per week, 198 class hours total)
Algebra (2 class hours per week)

	Class hours	Homework
1. Limits	6	3
2. Progressions	14	7

		Class hours	Homework
3.	Exponential and logarithmic functions. Logarithms	40	20
4.	Practical exercises in calculations with the slide rule	6	—
	Total	66	30

Geometry (2 class hours per week)

		Class hours	Homework
1.	Regular polygons	12	6
2.	Length of circumference and area of circle	10	5
3.	Solid geometry	40	20
4.	Practical exercises	4	—
	Total	66	31

Trigonometry (2 class hours per week)

		Class hours	Homework
1.	Trigonometric functions of an arbitrary angle	10	5
2.	Algebraic relationships between the trigonometric functions of the same angle	16	8
3.	Trigonometric functions of a numerical argument	16	8
4.	Addition theorem and its corollaries	24	12
	Total	66	33

Grade X (age 16–17)

(6 class hours per week, 198 class hours total)

Algebra (2 class hours per week)

		Class hours	Homework
1.	Combinations and Newton's binomial theorem	12	6
2.	Complex numbers	12	6
3.	Inequalities	22	11
4.	Equations of higher degrees	12	6
5.	Review	8	4
	Total	66	33

Geometry (2 class hours per week)

		Class hours	Homework
1.	Polyhedra	28	14
2.	Solids of revolution	20	10
3.	Review and solving problems	18	9
	Total	66	33

Trigonometry (2 class hours per week)

		Class hours	Homework
1.	Solving scalene triangles	18	9
2.	Inverse trigonometric functions	14	7
3.	Trigonometric equations	16	8
4.	Practical exercises on location	6	—

5. Review of trigonometry and solving problems in trigonometry and geometry with applications of trigonometry	12	6
Total	66	30

The curriculum outlined above was a result of many changes, of which the most important have been introduced over the last four years. To fulfill the directives of the Twentieth Congress of the Communist Party regarding the transition from a compulsory seven-year school to general compulsory ten-year schooling and the further development of polytechnical instruction, Soviet educational authorities have reviewed the content of the courses and methods of instruction in the existing school. As mentioned before, the aims of Soviet general education are twofold: there must be sufficient preparation for those going on to higher institutions; there must also be supplied the knowledge and information which are indispensable for those going directly into professional activity upon graduation from the ten-year schools. With these goals in mind studies were undertaken to determine the basic shortcomings of mathematical education in Soviet secondary schools prior to 1956.

Because the great majority of graduates of the secondary schools immediately enter into practical work, one conclusion was that mathematical teaching had put too much emphasis on verbal book knowledge at the expense of practical mathematical knowledge. To correct this and other deficiencies, and to improve the teaching of mathematics in accordance with the new requirements of the Twentieth Congress, the Ministry of Education endeavored to change the curriculum. These changes were developed at the Academy of Pedagogical Sciences of the RSFSR with the active help of research mathematicians, methods specialists, and outstanding teachers.

The revised curriculum in the general-education schools was to be introduced in the following stages:

(1) in fifth-grade arithmetic in 1954–55;
(2) in sixth-, seventh-, and eighth-grade algebra and geometry in 1956–57;
(3) in ninth-grade algebra, geometry, and trigonometry in 1957–58;
(4) by 1958–59 the new program was to have been effected throughout all grades of the ten-year school.

Some of the more difficult topics in courses have been transferred from earlier grades to the next higher grades in order to lessen the amount of work required of the student, which was considered too severe. For example, the topic "Percentage" was transferred from the fifth grade to the sixth, "Factoring of polynomials" from the sixth to the seventh grade, "Extraction of the square roots of numbers" from the seventh to the eighth grade, etc.

The revised program shortened the size and content of such topics as "Solution of oblique triangles," "Trigonometric equations," "Inverse trigonometric functions," and others. Problems which required highly artificial methods or "tricks" for solution, or which demanded cumbersome and complicated computations and transformations, were eliminated whenever possible, since they had little educational importance, required too much work on the part of the students, and tended to undermine students' confidence in their own abilities.

This program greatly emphasizes functional relations, trigonometric functions of a numerical argument, and the application of trigonometry to the solution of practical problems in physics, astronomy, and technology. Increased significance has also been given to measurements, preparation of models, and drawing of

graphs and diagrams; the use of instruments of measurement, tables, and drawing tools is now stressed.

Khrushchev's proposals for changing the educational system are being elaborated by the Academy of Pedagogical Sciences of the RSFSR. A plan for the curriculum of the projected general-education eight-year school (which represents the compulsory, first stage of secondary education in the Soviet Union) has already been published in "Sovetskaya Pedagogika," January 1959.

The class hours assigned to mathematics in this plan are listed below. This will amount to 19.2 per cent of the total number of class hours to be spent in the eight-year school. The academic year will consist of 34 weeks in grades I–III and 35 weeks in grades IV–VIII.

Other Communist countries

The general-education schools in the other Communist countries are not exact replicas of the Soviet model. Neither the age of admission nor the length of the compulsory education are everywhere the same. In Czechoslovakia, for example, the age of admission to the schools is six, and compulsory education runs for eight years. Throughout eastern Europe,[8] however, there are eleven-year schools which correspond to the Soviet ten-year school. In these schools, the mathematics curriculum, at least for the present, does differ from that in use in the Soviet Union. The mathematics courses in the higher grades of the general schools in Romania,[9] for instance, include topics such as: elements of analytic geometry, differential and integral calculus, and higher algebra. These topics are not part of the Soviet curriculum.[10]

Russian influence, and possibly also Russian direction, show themselves in the emphasis placed on mathematics and the increased amount of time devoted to its study. The growing importance of mathematics is demonstrated by the fact that now, in the eleven-year schools of Czechoslovakia,[11] *20.5 per cent* of the whole school instruction time is devoted to mathematics, whereas prior to World War II only *13.5 per cent* was so allocated.

The distribution of the teaching of mathematics in the eleven-year schools of Czechoslovakia during the academic year 1955–56 is given in Table 1 on the following page.

In Table 2, similar charts for the Soviet Union, Poland, and Bulgaria are compared.[12]

Poland

The available literature shows that Polish educational authorities have complete freedom to experiment in their mathematics programs. General-education schools in Poland have eleven grades, of which the first seven are compulsory. The program in mathematics—especially in the higher grades—differs significantly from the corresponding one in Russia in regard

Grade	I	II	III	IV	V	VI	VII	VIII	Total
Class hours per week	6	6	6	6	6	6	5–6	5	46.5
Class hours per year	204	204	204	210	210	210	192	175	1609

[8] East Germany is an exception. It has a twelve-year secondary school consisting of the "Grund-Schule" for grades I–VIII and the "Ober-Schule" for grades IX–XII. Beginning with the academic year 1956–57 East Germany established a new, parallel secondary school called the "Mittelschule," which is a ten-year school and is modeled after the Soviet school.

[9] Sovetskaya Pedagogika (Soviet Pedagogy) 4, 1957, pp. 93–100.

[10] However the Soviet mathematics syllabus for the academic year 1958–59 was to include, for the first time, the following topic in the tenth grade: "Functions and their investigation. Derivative."

[11] Matematika v shkole (Mathematics in School) 6, 1956, pp. 64–67.

[12] Soviet Union: academic year 1957–58; Poland: 1957–58; Bulgaria: 1951–52.

TABLE 1

GRADE	I	II	III	IV	V	VI	VII	VIII	IX	X	XI
Number of class hours per week	4	6	6	6	7	7	6	6	6	5	5

TABLE 2

GRADE		VI	VII	VIII	IX	X	XI	Total
Soviet Union	Class hours per week	6	6	6	6	6	—	30
	Class hours per year	198	198	198	198	198	—	990
Poland	Class hours per week	6	5	5	4	4	4	28
	Class hours per year	192	160	160	128	128	128	896
Bulgaria[13]	Class hours per week	5	5	6	4	5	4	29
	Class hours per year	165	165	198	132	165	128	953

to course content as well as order and treatment of the topics.

Polish schools teach arithmetic in the first six grades. Geometry is given in three stages. The first stage (grades III–V) offers propaedeutic information on geometric figures and measurement. This material is correlated to concurrent courses in arithmetic. The second and third stages are concerned with synthetic geometry. The second stage (grades VI–VII) covers elementary geometry; its primary purpose is to give the graduate of the seven-year school a fairly adequate training in this field. The third stage (grades VIII–XI) consists of a more extended treatment of plane and solid geometry. In addition to this, algebra is taught from the seventh to the eleventh grade and trigonometry in the tenth and eleventh grades.

Following is a description of the mathematics program of the final six years of the Polish eleven-year school used in 1957–58:[14]

GRADE VI (AGE 12–13)
(6 class hours per week)

Arithmetic (128 class hours)

1. Decimal numbers
2. Ratio and proportionality
3. Computing interest

Geometry (64 class hours)

1. Fundamental geometric figures
2. Axial-symmetry
3. Circle
4. Congruence of triangles
5. Parallel straight lines

GRADE VII (AGE 13–14)
(5 class hours per week)

Algebra (80 class hours)

1. Signed numbers

[13] Matematika v shkole 5, 1955, pp. 60–73.

[14] Matematika v shkole 3, 1957, pp. 71–74.

2. Algebraic expressions
3. Equations of the first degree with number-coefficients
4. The concept of function

Geometry (80 class hours)

1. Figures in circle
2. Similarity of figures
3. Metric properties of figures
4. Introduction to solid geometry

Grade VIII (age 14–15)

(5 class hours per week)

Algebra (96 class hours)

1. Fundamental algebraic transformations
2. Factoring polynomials
3. Algebraic fractions
4. Systems of equations of the first degree

Geometry (Deductive course) (64 class hours)

1. Fundamental geometric figures
2. Parallel straight lines
3. Relationships between the sides and the angles of a triangle
4. Straight lines and circles
5. Geometric loci, angles, and polygons in relation to a circle

Grade IX (age 15–16)

(4 class hours per week)

Algebra (70 class hours)

1. Irrational numbers and irrational expressions
2. Quadratic equations
3. Systems of equations of the second degree with two unknowns

Geometry (58 class hours)

1. Measurement of geometric quantities
2. Similar figures
3. Numerical relationships between the elements of geometric figures
4. The length of the circumference and the area of a circle

Grade X (age 16–17)

(4 class hours per week)

Algebra (45 class hours)

1. Powers with real number exponents
2. Exponential function
3. Logarithmic function
4. Number sequences

Geometry (53 class hours)

1. Points, straight lines, and planes in space
2. Angles in space
3. Parallel projection on a plane
4. Areas of surfaces and volumes of polyhedra

Trigonometry (30 class hours)

1. Trigonometric functions
2. Right triangles

Grade XI (age 17–18)

(4 class hours per week)

Algebra (45 class hours)

1. Inequalities
2. Extension of knowledge of equations and inequalities

Geometry (20 class hours)

Solids of revolution

Trigonometry (63 class hours)

1. Fundamental trigonometric relationships
2. Relationships between the elements of a triangle and applications

Red China

According to Soviet sources,[15] a uniform national system of education was achieved throughout China in 1956. The government reformed school instruction and adopted new plans and programs. The original plan in 1951 called for an eleven-year complete elementary and secondary education, which included a compulsory five-year elementary school. For various reasons, the elementary school is now tem-

[15] Sovetskaya Pedagogika 2, 1957, pp. 106–116, Matematika v shkole 3, 1957, pp. 77–80.

TABLE 3

Mathematics in the middle schools of the Chinese People's Republic	Junior Middle School						Senior Middle School						
	I		II		III		I		II		III		
	Semesters												
	1	2	3	4	5	6	1	2	3	4	5	6	
	Number of class hours per week												Total
Arithmetic	7	7											252
Algebra			3	3	3	3	4	3	2	2	2	2	463
Geometry			2	2	3	3	2	3	2	2	2	2	356
Trigonometry									2	2	2	2	142

porarily maintained as a six-year school. Students are admitted to the first grade at the age of seven.

The chart below shows the number of class hours devoted to mathematics in the Chinese elementary school in 1955–56 (arithmetic with special attention to the mastering of the abacus):

GRADE	I	II	III	IV	V	VI
Class hours per week	6	6	6	7	6	5

This amounts to 1224 class hours, representing *24.3 per cent* of the entire elementary school time.

The general-education secondary school in China consists of two levels. The first level is a three-year school called "junior middle school," admitting students at the age of twelve. The second level is also a three-year school called "senior middle school," admitting its students at the age of fifteen. A graduate of an elementary school is admitted to the junior middle school solely on the basis of a severe competitive examination. In 1954–55 there were five applicants for each opening.

In the curriculum of the middle schools mathematics again occupies a very important place, amounting to *18 per cent* of the whole instruction time. This came to a total of 1213 class hours in the academic year 1955–56, distributed as indicated in Table 3.

An additional 107 class hours were devoted to technical drawing.

The mathematics program of the Chinese middle schools is almost identical with that of grades V–X in the Soviet schools. However, on the average the Chinese devote 10 per cent more class hours to algebra, geometry, and trigonometry than do the Russians.

It is worth adding that the Chinese teachers of mathematics watch Soviet achievements with close attention. All fundamental Russian mathematical textbooks and several handbooks and monographs have been translated into Chinese.

In the Soviet Union, education in general (and technical education in particular) is one of the safest avenues to preferment and position. The student is constantly made aware of the importance of mathematics for his future profession. And since the entire educational system is selective—only the best fitted are admitted to higher schools—the students are highly

motivated to study mathematics, for which ample teaching time is made available. No one ever becomes a physician, a sociologist, a schoolteacher, or an army officer without going through the required ten years of mathematical training. Even the millions of children who do not graduate from the ten-year school—each and every one of them has had, besides a full training in arithmetic, at least 330 class hours of instruction in algebra and geometry; and every student who graduates from the projected, compulsory eight-year school will have had more than 500 class hours of algebra and geometry.

Russian educational authorities follow and study carefully developments in American education. They are familiar with the inadequacies and shortcomings of the mathematics curriculum which are of such concern to us. The March–April, 1957 issue of a Soviet mathematical periodical offers, with obvious satisfaction, the following quotation from the American journal *Industrial Arts and Vocational Education* (June, 1955):[16]

> If high school graduates desire to enter skilled trades and the technical professions, it will be necessary for them to pursue subjects in high school which will permit them to enter training programs for these occupations. The facts, however, are very discouraging. Last year, according to one of our chief industrial recruiters, only 1.4 per cent of high school students took solid geometry, only 1.6 per cent took trigonometry, 0.5 per cent took college algebra—in fact, less than 20 per cent of our high school population pursued mathematics beyond arithmetic. Less than 7.6 per cent took chemistry and only 4.3 per cent selected physics. In other words, the demand for skilled workers, scientists, and technicians is increasing while the potential supply is decreasing. For many occupations, the need for a good mathematical foundation was never greater than it is today.

[16] From an article "Trends on the Educational Horizon" by Heber A. Sotzin (p. 187), Matematika v shkole 2, 1957, p. 67.

If the development of the human capacity for reflection is the essence of education and consequently the essential task of educational institutions, the mere accumulation of information is not education. Indoctrination in even the noblest ideas and ideals is not education. Nor is the mastering of vocational skill true education if all that it involves is the training of hands to perform a task efficiently.—Soviet Professional Manpower, *National Science Foundation*, 1955, *p.* 406.

The science of physics does not only give us (mathematicians) an opportunity to solve problems, but helps us also to discover the means of solving them, and it does this in two ways: it leads us to anticipate the solution and suggests suitable lines of argument.—*Henri Poincaré*

Providing for the slow learner in the junior high school

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While planning work for the slow learner, we must keep in mind some psychological facts about these pupils if the program is to be effective.

THE CONCEPTS AND IDEAS of "modern" mathematics are on everyone's mind. The curriculum for the bright college-bound student is under close examination by many groups of leading educators and the development of a modern, up-to-date curriculum for this group in the near future seems inevitable. While this all-important phase of the mathematics program is being studied, let us occasionally pause and reflect upon the lot of the slow learner who normally terminates his formal instruction in mathematics by the end of the junior high school years. What are the mathematical needs of this group and how can we best provide for these needs? This article will attempt to pose many questions and suggest answers to at least some of these.

It will be helpful to begin by an examination of the psychology of junior high school youth in general and then relate this to the slow learner in particular. Next, a variety of procedures for handling the slow learner will be considered. Finally, with this background, the mathematics curriculum which should be provided for this type of student will be explored.

The emphasis on the teaching of mathematics has always been on the logical systematic development of subject matter, to the neglect, all too often, of the psychological basis. In the junior high school where the mathematics taught is often not sequential in nature, the psychological emphasis is perhaps the more important one to consider. What are some of these psychological considerations? As a start, consider the following list of characteristics and needs of all junior high school youth:

1. In the first place, these are the years of rapid, uneven growth and there is a need for both teachers and students alike to understand this growth and to realize that it is both variable and individual in nature.
2. Youngsters of this age, continuing a drive which begins at birth, seek personal independence from both parents and teachers.
3. They seek peer acceptance and have a need to belong to some social group.
4. They are insecure, primarily due to the tremendous physical changes taking place at this age, and crave security and success.
5. They want recognition, approval, status.
6. Their interests change rapidly; they crave new experiences while at the same time longing for the security of the old.

These characteristics are familiar to any person who has taught in the junior high school. Now let us see how these items may be used as a psychological basis for the mathematics curriculum, with particular application to the slow learner.

First of all, it is important to build on earlier related learnings so as to lead to a sense of security. A careful gradation of

difficulty is important. Too radical an increase leads to trial-and-error behavior in order to find some scheme to succeed and thus gain security.

Whereas all students crave security and have a need to succeed, the slow learner is especially vulnerable in this respect. Years of consistent failure in the early grades make him prey for any sort of meaningless trial-and-error scheme just to get an answer and satisfy the demands of the teacher.

It is important that we make an effort to motivate the student. At ages thirteen to fifteen there is resistance to learning unless the subject matter is of interest to the student and meaningful to him; again, the slow learner can become especially resistant unless his interest, dulled by years of failure, is aroused. Specific ways of arousing this interest will be considered later. It can be said, however, that we cannot interest these youth by deluging them with huge quantities of social applications with the excuse that these are of practical value. Some may be; most are not. We need to present honest-to-goodness mathematics to them in an organized and systematic structure, but in a manner which will evoke their interest and natural curiosity.

Junior high school youth in general, and slow learners in particular, are eager to grasp and adopt patterns of work providing them with security and independence. They thus are prone to learn tricks and meaningless manipulations in order to achieve success. Students given numerous problems to solve by rote tend to repeat these methods in tasks that can be solved by more direct means. This rigidity of thinking, following of patterns, does not place a stress on thinking or on basic concepts; rather it places a premium on obtaining answers.

To summarize briefly: the slow learner in the junior high school has the same characteristics as other pupils of the same age; the same basic needs and interests. However, more than the average child, he needs to be given the chance to experience success and approval; more than the others he needs to feel that he is a member of the group with a contribution to make; he needs status; his confidence must be re-established, his interest stimulated, his attitude towards mathematics made favorable, his ego flattered.

At the junior high school level of instruction we receive students who have endured six years of study of arithmetic. It is fair to assume that they have been in contact with the fundamental operations of addition, subtraction, multiplication, and division of whole numbers, fractions, and decimals; they should have some understanding of our number system; they should be able to analyze and solve simple problems; and they should have a knowledge of certain basic geometric forms.

We find, however, that many students come to us in the seventh grade with deficiencies that place them as far down as the second- or third-grade level of ability in arithmetic. With the modern concept of universal social promotion, these students reach junior high school and are not at all equipped to do the conventional work of these grades.

Let us take a moment to look at some administrative procedures for handling them. To begin with, ability grouping, once considered to be quite undemocratic, is now increasing rapidly throughout the country. This appears to be one of the easiest means of handling the problem. Where this is not a feasible solution it may be possible to try grouping within a classroom, but this is much easier said than done.

Remedial mathematics classes can produce astonishing results, but they are effective only if the number in each group is limited to a maximum of fifteen, preferably twelve, so that each child may be taught on an individual basis. Very often we find that a slow learner shows a great deal of improvement when time is taken to teach him basic concepts and the rationale of the fundamental computations that he

had heretofore considered only a series of tricks. Special mathematics clinics, similar to the reading clinics found in many schools, are also of value.

Incidentally, mere segregation of the slow learner does not imply that better learning will take place; rather it demands that the entire curriculum must be changed so as to allow the individual the opportunity to make the most of his capabilities.

It is appropriate now to set forth a number of specific suggestions which are important in guiding the learning activities of the slow student. Most of the items listed below apply to all youth; but whereas they are helpful to all youth, they are an absolute necessity for the slow learner.

1. Because of a very short attention span, the activities of the slow learner must be varied. Too often the new, inexperienced teacher neglects this item and frequently meets with disastrous results. Projects, laboratory work, board work, supervised study, recitation, and so on must be mixed in liberal dosages.
2. Concrete presentations must be emphasized. Laboratory techniques and manipulative materials are essential. The use of aids, models, films, and filmstrips can play an important part in the education of these youth.
3. An emphasis on practical applications is important. This does not imply that this is the only procedure for gaining the interest of, or for motivating, this group, but it is one possible means of so doing.
4. These students must be allowed to compete with themselves and their achievement should be measured in terms of individual growth. It makes no sense, and it frustrates both teacher and child alike, to try and evaluate students with many years of deficiencies in terms of a preconceived standard of what should be accomplished in grades seven, eight, or nine.
5. Topics must be taken up in spirals; not taught once and then forgotten.
6. Where possible, subject matter should be correlated with work in other classes. This helps develop a feeling of security on the part of the student. Thus when the science teacher is studying the solar system, for example, we in the mathematics classroom can make scale drawings of the planets, graph the distances to the planets, and solve problems using data discussed in the science classroom.
7. Drill is essential, but it must be meaningful and not rote. It must be varied so that improper mental sets are not established. The slow learner, more so than others, must understand what he is doing.
8. Verbal materials in the text must be developed orally. We can not assume comprehension of reading; the slow learner is, more often than not, retarded in this area as well.
9. Frequent reviews are necessary.
10. In his need of security, the slow learner appreciates and does best in a situation where classroom management is routine.
11. Successful student materials should be exhibited to provide a feeling of success.
12. The final item concerns the procedure used to start the school year, whether it be in grade seven, eight, or nine. There is little doubt but that most slow learners in the junior high school are in dire need of a meaningful reteaching of arithmetic. As a matter of fact, this should probably be the fundamental work of these grades. On the other hand, experience indicates that beginning the seventh grade work with this much-needed review of arithmetic tends to be dull, deadening, and destructive of any interest that might otherwise be aroused.

This is not to imply that we should teach anything less than mathematics,

but merely that there are other aspects of the curriculum with which to begin a new course that are more appealing than the traditional review of arithmetic, no matter how great the need. A sample of several of these appealing topics would include such items as the use of other number bases, systems of numeration, ancient methods of computation, experimental probability, and elementary surveying.

Please do not misunderstand; we should not flood the curriculum of the slow learner with unusual or recreational material to entertain him. Rather we must make use of these "side roads" of mathematics to arouse his dulled interest and motivate him to a further study of mathematics.

Finally, let us examine the content of the curriculum for the slow learner in the junior high school. In the area of arithmetic we must accept the students as they come to us and permit them as much growth as possible. We can best accomplish this by a thorough and meaningful reteaching of arithmetic. Just routine drill on the same processes which caused these youngsters difficulty for six long years will not do. However, because of the added maturity of the seventh grader, many of the fundamental processes which caused him difficulty in earlier years now prove to be within his grasp, if they are properly retaught.

The stage must be set for this reteaching by motivating or interesting the student in the subject of mathematics; by rekindling the early love of mathematics that all children naturally seem to have before it is destroyed somewhere along the line.

Social applications are important, but we must not work these to death. The average seventh grader, let alone the slow learner, has very little interest in a study of stocks and bonds, budgets, or taxation. This is not to say that there is no place in the curriculum for social applications. The use of baseball statistics in relation to work on per cent may capture his imagination. Life insurance may be too remote a topic for his interest, but the formation of a book insurance company within the classroom to insure against lost books not only may be of practical value but may prove to be a motivating device as well.

A great deal of review of the processes of arithmetic can be developed by means of a study of statistics and probability. Now this may sound strange because this happens to be a topic which the Commission on Mathematics is recommending for the senior year of study for the college preparatory program. However, a unit on this topic, in the junior high school, for the slow learner, can prove to be one of the most interesting parts of the years' work. We will not use, of course, the unit prepared by the Commission, although this is excellent background material for the teacher. Rather we may employ the technique wherein some of the basic concepts of probability and statistics are developed by means of an informal, laboratory, inductive approach. Thus the student gathers and analyzes real data by means of penny-tossing experiments, tossing of dice, taking samples of chips from a box, etc.

Such a unit, incidentally, allows the study of graphs, such as those showing distributions of heights, weights, and even the age of puberty for boys and girls. All of this allows the youngster to develop a better understanding of himself as an adolescent.

The curriculum in geometry for the slow learner in the junior high school should be developed as an experimental, informal, investigative approach, with the laboratory method of teaching as the predominant procedure to be followed. A few examples should serve to establish the point of view being expressed here:

I. Informal deduction can be developed by actual measurement, paper folding, etc. Laboratory experiments can be set up whereby the student is asked to form generalizations on the basis of a number of specific instances. For example, a sheet can be provided with a series of parallelo-

grams drawn. The student is asked to draw the diagonals in each figure, measure the segments of these, and deduct that the diagonals bisect each other. He is led to this conclusion by means of a carefully described process on a laboratory sheet such as the following:

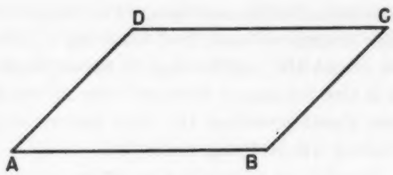

Figure 1

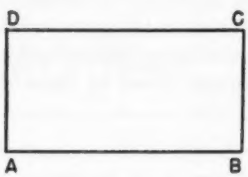

Figure 2

Figure 3

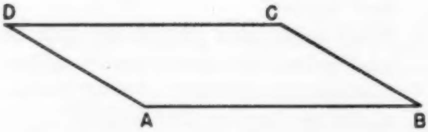

Figure 4

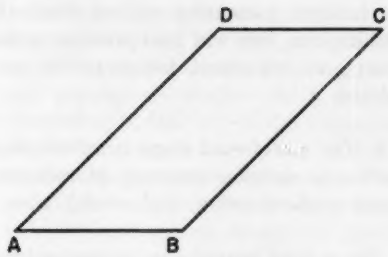

Figure 5

The following experiment is based on the five parallelograms drawn above.

1. Draw the diagonals, AC and BD, in each figure. Call the point of intersection of the diagonals M.
2. Measure AM, MC, BM, and MD in each figure and record your data in the following chart:

Figure	AM	MC	BM	MD
1				
2				
3				
4				
5				

3. Compare the lengths of AM and MC in each figure.
4. Compare the lengths of BM and MD in each figure.
5. What do you conclude about the diagonals of a parallelogram?
6. Test your conclusion by constructing a parallelogram, drawing the diagonals, and measuring each of the segments.
7. Test your conclusion by repeating the previous exercise with a four-sided figure which is *not* a parallelogram.

II. Elementary surveying enables the student to see some practical applications of the geometry he has studied, while at the same time it offers training in the areas of estimation, space relations, scale drawings, etc. Each student can make a simple clinometer or hypsometer and use the device

for indirect measuring out of doors. If nothing else, this will lend prestige to the group, an important factor to be considered.

III. The use of road maps is an excellent device to develop concepts of measurement, scale drawing, and co-ordinates.

These brief remarks on geometry have been made to indicate a point of view. We can teach the slow learner to appreciate the beauty of geometric form, to develop concepts of space relationships, to learn certain basic facts, and to develop skill in the use of the basic tools of geometry. We can do much more than just allow him to draw geometric designs as is so often done with this type of student. He too tires of drawing designs!

The emphasis on algebra should probably be relatively a minor one. Algebra can be introduced in the unit on informal geometry, using it naturally in connection with formulas for areas, perimeters, and volumes of geometric figures. This does not imply that algebra cannot be taught to the slow learners. Unfortunately it is taught to them all too often as the same series of meaningless manipulations that they had been asked to master in arithmetic. They can be taught the mechanics of algebra, and with the guidance of an enthusiastic teacher can even be taught to enjoy these mechanics, but this is not a worth-while activity.

On the other hand, algebra can be taught as generalized arithmetic which will help reinforce knowledge of the basic fundamentals; this is very worth-while. One can, for example, develop a real understanding of the addition of fractions in arithmetic by generalizing this with algebra.

It is also of value to present to the slow learner those topics of algebra which arise in realistic situations. Thus we can relate the so-called three cases of per cent by means of a single formula: $P=br$. Other formulas which help explain certain physical phenomena can be introduced. Certainly formal algebraic drill on such topics as "Complex fractions" or "Equations" appears to be out of place for this group.

It's quite possible that this article has attempted to cover too broad an area and should have been limited to a much narrower field. On the other hand, it is not possible to intelligently discuss curriculum provisions for the slow learner in the junior high school without first knowing a little bit about the psychology of these youth. It is also necessary to have some information about handling the slow learner and guiding his learning activities.

Finally, an attempt was made to point out specific areas in the curriculum where attention could be given to the slow learner by providing him with a meaningful, structured approach to mathematics, but with emphasis on laboratory techniques.

There are many unanswered questions which remain; some of these are listed below.

1. What should be the content of a three-year program of mathematics at the junior high school level for the slow learner? How much emphasis should be given to practical applications?
2. What procedures are available to properly motivate and develop favorable attitudes among this group?
3. What teaching methods prove to be most successful with the slow learner? What evidence is available concerning the use of aids and laboratory techniques with this group?
4. What constitutes an adequate program of remedial arithmetic in the junior high school and how shall such a program be organized?
5. What is the role of evaluation for the slow learner?
6. What topics and procedures may be used to lend prestige to classes of slow learners, to so-called general mathematics sections in many schools?
7. Finally, are any of the so-called mod-

ern topics suitable for the slow learner? Attempts are being made by some individuals to adapt certain topics of modern mathematics to the below-average student. More such experimentation is needed. Can, for example, the slow learner also profit by a discussion of the elements of sets?

The preceding list of questions has been classified as unanswered items, despite the fact that this article has attempted to provide some of the answers. What is implied is that careful studies are needed concerning the interests, abilities, and teaching methods suitable for this large segment of our population—studies which might be equivalent to that which the Commission on Mathematics is attempting to conduct for the college preparatory student. This is no simple task, but the results would be valuable and eagerly sought by teachers throughout the country. In our rush to provide for the gifted, let us not forget the slow learners—there are so many more of them!

What's new?

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Angle trisection—an example of "undepartmentalized" mathematics

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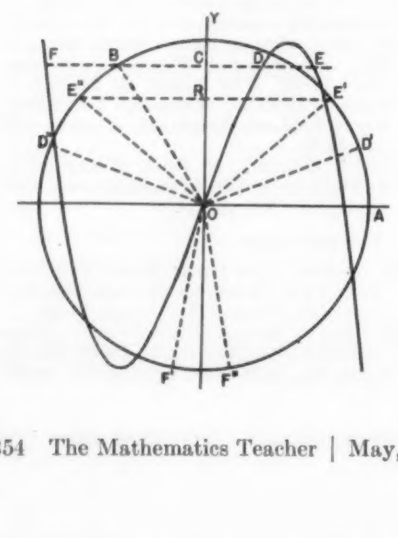
An old, old problem—but the emphasis is different, from the usual presentation.

WITH THE MODERN emphasis on integration in mathematics, teachers might find it profitable to consider a method for trisecting an angle that is a fine illustration of the correlation of algebra, geometry, trigonometry, and analytic geometry. In short, the formula $\sin 3A = 3 \sin A - 4 \sin^3 A$ is expressed as $y = 3x - 4x^3$ and graphed.

Let y represent $\sin 3A$, i.e., the sine of the angle to be trisected, and x represent $\sin A$ or the sine of one third of the original angle. Then a given value of y determines three values of x, from one of which the required angle may be obtained.

The accompanying graph shows the procedure. Place the angle to be trisected in ordinary position with vertex at O, initial side OA, e.g., angle AOB. From point B where the terminal side of the angle meets the unit circle, draw BC perpendicular to OY, extending it to cut the graph of the cubic at D, E, and F. Lay off on OY a distance OR equal to CE (E is chosen because the obtuse angle AOB has the same value for its sine as its supplementary acute angle, and therefore the second possible value for the sine A is used), and construct by means of the unit circle the angle having OR as its sine. Angle AOE' is therefore one third of angle AOB.

Figure 1



In graphing the cubic the following points should be noted:

1. The curve is symmetric about the origin. This may be observed from either the algebraic or the trigonometric form.
2. It has intercepts $0, \dfrac{\pm\sqrt{3}}{2}$ on the x axis.
3. Its range should be from -1 to $+1$ for both x and y, to correspond to the range of values for the sine of an angle.
4. It has a relative maximum at $x=\frac{1}{2}$, $y=1$ and a relative minimum at $x=-\frac{1}{2}$, $y=-1$. This may be shown by writing the equation as $y=1-(2x-1)^2(x+1)$ and also as $y=-1+(2x+1)^2(1-x)$.

The method correlates the following topics:

FROM ALGEBRA AND ANALYTIC GEOMETRY,

it utilizes the Cartesian co-ordinate system in graphing the cubic $y=3x-4x^3$ and the unit circle $x^2+y^2=1$.

FROM GEOMETRY,

it applies the drawing of perpendicular lines, a circle, and the construction of angle AOE'.

FROM TRIGONOMETRY,

it requires the development of the formula for $\sin 3A$, which embraces the use of $\sin 2A$, $\sin (A+B)$, and $\sin^2 A+\cos^2 A=1$.

it uses also the unit circle in two ways: first, given angle $3A$ to determine the representation of its sine, and secondly, given the value of the sine A to construct angle A.

it emphasizes the fact that angles in the second quadrant have the same value for the sine as their supplements (this should be extended to angles terminating in the third and the fourth quadrants).

it establishes that angles of any size whatever, even greater than a complete revolution, might also be trisected by the proper determination of the correct value for x, and the corresponding correct value for angle A.

The following table illustrates the choice to be made, based on the assumption that angle AOB is 120°. From the table the proper point is chosen to determine the terminal side of angle A. The points then repeat themselves at intervals of 3 revolutions.

Every teacher should impress on his students that this is not a solution of the classical problem, since methods other than the use of straightedge and dividers are used, notably the freehand drawing of the cubic.

If $3A$ is:	60°	120°	420°	480°	780°	840°	1140°	1200°
A is :	20°	40°	140°	160°	260°	280°	380°	400°
Point	D'	E'	E''	D''	F'	F''	D'	E'

The circle is the first, the most simple, and the most perfect figure.—*Proclus*

Lo cerchio è perfettissima figura.—*Dante*

Letter to the editors

Dear Editor:

In regard to Professor Oesterle's article in the October issue of THE MATHEMATICS TEACHER and my letter to the editor published in the February issue, may I say that my interpretation of the statement found in Professor Oesterle's article and referred to in my letter is erroneous.

I hope you will be able to find a place for this letter in an early issue of THE MATHEMATICS TEACHER, thereby correcting, insofar as possible, any misinterpretation of Professor Oesterle's excellent article that may have been caused by my letter.

Sincerely yours,

JOHN BARRY LOVE
Asst. Professor of Mathematics
Eastern Baptist College
St. David, Pennsylvania

Editorial Note: And the Editor must share the blame for too quickly passing a comment that looked like an editorial error.

One plus one equals toon

FRANKLIN S. McFEELY, *Montana State College, Bozeman, Montana.*

Names for binary numbers are proposed. The names are similar to those for decimal numbers with the same representation.

MANY OF US HAVE, at one time or another, amused ourselves by performing arithmetic with numbers whose radix (base) is other than ten. With the appearance of automatic digital computers, the use of at least one radix other than ten, viz., two, has become of some importance. Numbers with this radix are often called *binary numbers*. Since only two symbols, 0 and 1, are required to express binary numbers, rules for addition are simple, being expressed by fairly simple devices in basic electrical and electronic circuitry (flip-flops, change of polarity, etc.). Most digital computers perform addition using binary numbers, although, of course, input and output data may be in decimal form.

To keep pace with their new importance, it would be fitting to agree upon a terminology for binary numbers analogous to that for decimal numbers. It appears that there are two reasons for attacking this problem. The first is that such a terminology would facilitate the paper-and-pencil addition of binary numbers. The second relates to the desirability of associating magnitudes with binary numbers directly without the necessity of converting them to decimal numbers. This problem has been considered by Stern[1] and others.[2] The names proposed here are somewhat different from those put forward by Stern.

Once the rule $1+1=10$ is thoroughly in mind, addition of two binary numbers of two or more significant digits each is easily performed, and this ability thus allows us to add any number of binary numbers by performing successive additions of two numbers. In order to add more than two binary numbers directly, without reducing the sum to several sums of two numbers, and to perform the addition, thinking in binary numbers instead of in decimal numbers and then converting, names for the binary numbers would be helpful. The names below are proposed.

The "justification" for this terminology is: (1) The names, when spoken, indicate that the numbers are in radix two. Since the spelling "two" might appear awkward, the spelling "too" is used. (2) The names are somewhat similar to the names given decimal numbers having the same representation (but not magnitude, of course). Using this terminology, binary numbers are named in the same manner as decimal numbers, e.g., the binary number 1111 is called one toosand one toodred etooven. Counting proceeds one, toon, etooven, one

Binary representation	10	11	100	1000	1,000,000	1,000,000,000
Binary name	toon	etooven	toodred	toosand	mitoollion	bitoollion
Pronunciation	to͞on	ē-to͞o′-ven	to͞o′-dred	to͞o′-sand	mī-to͞ol′-lion	bī-to͞ol′-lion
Decimal magnitude	2	3	4	8	64	512

[1] Joshua Stern, "A System of Names for Binary Numbers," *Science*, CXXVIII (1958), 594–95.

[2] A. S. Householder and Joshua Stern, "Letters," *Science*, CXXVIII (1958), 1246, 1298. My colleague, Dr. Adrien Hess, called my attention to these references.

toodred, one toodred one, etc., and this is what we do when we add.

Now consider an example in addition. The procedure is as follows:

$$
\begin{array}{ccccc}
 & & 1 & 1 & 1 \\
 & & 1 & 0 & 1 \\
 & & 1 & 0 & 0 \\
 & & 1 & 1 & 1 \\
 & & 1 & 1 & 0 \\
\hline
1 & 1 & 1 & 0 & 1 \\
 & (3) & & (2) & (1)
\end{array}
$$

(1) In the ones' position we count one, toon, etooven (11), hence we write 1 and have 1 to carry. (2) In the toons' position, we have one (carry), toon, etooven, one toodred (100), hence we write 0 and have toon to carry. (3) In the toodreds' position we start at toon (carry), and proceed to etooven, one toodred, one toodred one, one toodred toon, one toodred etooven (111), which we write as shown. A little practice should make one quite articulate in this system and able to add binary numbers with facility. We can then proceed to subtraction, multiplication, and division. Finally, analogous to the decimal point, we may introduce a *toocimal* point. Names for binary numbers less than one are obtained by attaching a suffix to the name of the reciprocal of the number exactly as is done in the decimal system, i.e., 0.1 is "one-toonth"; 0.01 is "one-toodredth"; etc. This then gives us a set of names for binary numbers as complete as the set for decimal numbers, as far as representation is concerned.

Arithmetic—Early Nineteenth Century

"A gentleman a chaise did buy,
A horse and harness, too.
They cost the sum of three score pounds,
Upon my word 'tis true.
The harness came to half the horse,
The horse to twice the chaise,
And if you find the price of them,
Take them and go your ways."

Solutions to Problems in May 1959 issue, "Student Journal"

Doublets. The first player should write X in squares 8, 9. The figure is now symmetrical. The first player keeps it so. Whatever the second player does, the first player does the same thing on the other side. For example, if the second player fills squares 11, 12, the first player fills 5, 6. If the second player fills 1, 2, the first player fills 15, 16. In this way the first player is sure to win.

The Expanding Railroad. By the Pythagorean Theorem, $PQ^2 = 1.0003^2 - 1^2 = 0.00060009$. Taking the square root, we find $PQ = 0.0245$ approximately. So Q is 0.0245 of a mile above P. This is about 129 feet, much more than most people guess.

Problem 135. Dividing the Area. Starting from A, go 4 spaces to the right and $1\frac{3}{4}$ spaces up. Mark this point and join it to A by a straight line. Below this line we have a triangle of base 4 and height $1\frac{3}{4}$, also a single square. The area below is thus $\frac{1}{2} \times 4 \times 1\frac{3}{4} + 1 = 4\frac{1}{2}$, as required; there are 9 squares altogether.

Why Surprises Happen. Question 5. Yes. Question 7. Yes. Question 8. $n \cdot n + 8n + 15$. Question 9. $(n+4)(n+4) = n \cdot n + 8n + 16$. $(n+3)(n+5) = n \cdot n + 8n + 15$. The first expression is always bigger by 1 than the second.

Trisecting an angle

C. CARL ROBUSTO, *St. John's University, Brooklyn, New York.*

In view of the interest many of the "uninitiated" show in angle trisection, commentary on a proof that trisection is not possible, and presentation of the proof of impossibility, may serve a purpose.

FOUR CONSTRUCTION PROBLEMS—doubling the cube, squaring the circle, constructing the regular heptagon, and trisecting an angle—*using a straightedge and compass alone,* are the more common of the classical Greek problems of geometrical construction. Of these problems, trisecting a given angle is perhaps the most popular. Through the years many alleged proofs or solutions have been advanced for this unsolvable problem. Of course there are angles, such as 90°, 135°, 180°, and 360°, for which the trisection can be performed. To establish that the trisection of an-angle, using a straightedge and compass alone, is in general impossible, it is sufficient to show but one angle that cannot be trisected, since a valid general method would have to include every single case. Thus, the nonexistence of such a general method will be established if we can show, for example, that an angle of 60° cannot be trisected by a straightedge and compass alone.

A straight line can intersect a circle in two points only, and these points of intersection may be determined merely by the solution of a quadratic equation. Conversely, a quadratic equation may be solved by means of a straightedge and compass. Now, if we are given any angle, let us say equal to θ, the problem is to determine $(\theta/3)$. An algebraic equivalent of this problem may be obtained in various ways. A simple procedure is to consider an angle θ as given by its cosine, $\cos\theta=y$. The problem, then, is equivalent to that of finding the quantity, $x=\cos(\theta/3)$. From plane trigonometry the cosine of $(\theta/3)$ is related to that of θ by the equation

$$\cos\theta=y=4\cos^3(\theta/3)-3\cos(\theta/3). \quad (1)$$

In other words, the problem of trisecting the angle θ with the $\cos\theta=y$ amounts to constructing a solution of the cubic equation

$$y=4z^3-3z \text{ or } 4z^3-3z-y=0. \quad (2)$$

Now, let $\theta=60°$, so that $y=\cos 60°=\frac{1}{2}$. Equation (2) then becomes

$$8z^3-6z=1. \quad (3)$$

It is merely necessary now to show that this equation has no rational root, and the impossibility of trisecting the angle is established. Equation (3) has no rational root. Thus, the trisection of the angle is established as impossible, using a straightedge and compass alone. The first rigorous proof that an angle cannot be trisected by straightedge and compass alone was presented in 1837 by P. L. Wantzel.

The proof that the general angle cannot be trisected with a straightedge and compass alone is true only when the straightedge is considered as an instrument for drawing a straight line through any two given points and nothing else. By permitting other uses of the straightedge, the totality of possible constructions may be greatly extended. The restriction to the straightedge and compass alone dates back to antiquity, although the Greeks themselves did not hesitate to use other devices.

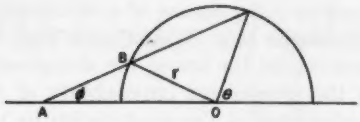


Figure 1

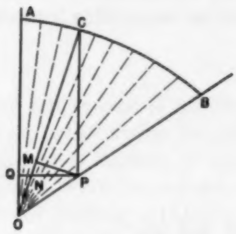


Figure 2

The following procedure for trisecting an angle, found among the works of Archimedes, is a good illustration.

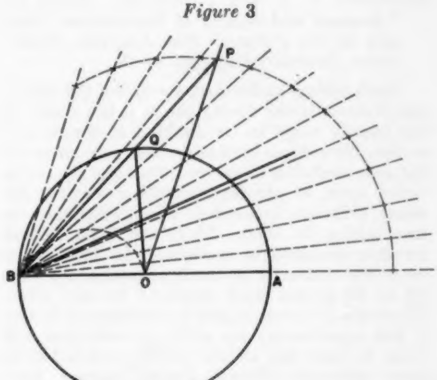
Let an arbitrary angle θ be given, as shown in Figure 1. Extend the base of the angle to the left, and describe a semicircle with O as center and arbitrary radius r. Now mark two points on the straightedge, A and B, such that $AB=r$. Keeping the point B on the semicircle, move the straightedge into the position where A lies on the extended base of the angle θ, while the straightedge passes through the intersection of the terminal side of the angle θ with the semicircle about O. Then, with the straightedge in this position, draw a straight line making an angle ϕ with the extended base of the original angle θ. This construction actually yields $\phi=(\theta/3)$.

A method of trisecting an angle approximately, in fact with a remarkably small error, attributed to Nicomedes, is as follows. Let AOB be the given angle. Through a point P on OB, draw, a line perpendicular to OA such that it meets AO at Q. Now draw a series of straight lines through O and lay off on them, from the points where they intersect PQ, a distance equal to $2(OP)$. If the points now determined are connected by straight lines, we determine approximately the locus of what is regarded as the "Conchoid of Nicomedes." To trisect the angle, we erect a perpendicular from P which meets the conchoid at the point C. As shown in Figure 2, it follows then that the angle AOC equals one-third of the angle AOB.

As a proof, draw the construction line PM. Then, $\sin OCP=PN/NC=PN/2(OP)$; $\cos MPQ=\cos OCP=PM/PN$; $\sin MOP=PM/OP$. From these equations we obtain: $\sin MOP=2(\sin OCP\cdot\cos OCP)=\sin 2(OCP)$. It follows, then, that angle $MOP=2(OCP)$. Since angle AOC $=$ angle OCP, the construction is established.

The following interesting geometrical construction will also trisect an angle. Describe a circle with center at O and a diameter, the extremities of which are B and A. From B, draw a series of chords extending through the circumference of the circle. On each chord, on both sides of the circumference, lay off a distance which is just equal to the radius of the circle. The locus of these points is called the "Limacon of Pascal." Now, as shown in Figure 3, upon OA as base with vertex at O, construct the angle to be trisected. Extend the side of the angle until it intersects

Figure 3



the limacon at the point P and draw BP. Let the point where BP intersects the circle be called Q. Now, $PQ=QO=OB$, which is the radius, and angle PBA is two-thirds of the angle POA. The trisection is completed by bisecting angle PBA.

The problem of trisecting an angle, as originally proposed by the Greeks, was to trisect an angle rigorously, without theoretical error, by means of a straightedge and compass only. Misunderstanding the restriction on the use of the straightedge and the consequent impossibility of accomplishment of the construction by use of the compass and *unmarked* straightedge leads people to attempt a thing that has been known as impossible for more than a century.

When you have satisfied yourself that the theorem is true, you start proving it.—The Traditional Mathematics Professor.

Have you read?

BANCROFT, T. A. and FUTRANSKY, DAVID L. "Demand and Supply of Statisticians Now and in the Future," *The American Statistician*, February 1958, pp. 20–24.

Both college and industry will feel the shortage of statisticians during the next ten years. If the supply were to be doubled it would not suffice. New departments as well as courses in the area are being developed. Colleges are being called upon to provide consulting services for many different industries. The government is also taking its share. There are 2100 federal services classified as statistical. In 1957 there were 300 vacancies in these positions and only 40 to 50 people were prepared to take over. There are all levels of jobs in statistics; it is also a fine opportunity for girls. I think you will want to read this article and recommend it to your students.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

BRYDEGAARD, MARGUERITE. "Expanding Horizons in Building Mathematics Competency," *Childhood Education*, December 1958, pp. 162–66.

The author points out that machines now do the "errand boy mathematics"; therefore, our education calls for a broader horizon. We must be concerned about quantity, size, order, and position. Teachers must fan the flame of curiosity held by the student.

Someone must lend a helping hand to the teachers also.

Action research in the field is needed in the areas of understanding, identifying facts about numbers, evaluating, building appreciation, and expanding the "how" to teach. The real power of mathematics must be felt from the kindergarten through the Ph.D. I think this article will confirm this feeling.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Mechanics of orbiting

M. V. LANDON, *Nasson College, Springvale, Maine.*
A timely article for mathematics classes in the high school.

THE BASIC EQUATIONS of orbiting for the satellites are easily within the grasp of the secondary-school student of algebra and trigonometry. While the refined equations, necessary for putting a satellite in orbit, are beyond the student of such elementary mathematics, the basic equations give excellent insight into the mechanics of orbiting.

The mechanics of orbiting are based on Newton's three laws:

1. A body at rest or in motion will continue at rest or in motion along a straight line at a uniform velocity, unless acted upon by an outside force.
2. If an unbalanced force acts on a body, the acceleration produced will be equal to the force divided by the mass of the body.
3. To every action there is an equal and opposite reaction.

Indeed, Newton himself worked out the mechanics of orbiting over two centuries ago and suggested launching man-made satellites from the tops of mountains, using big guns. Unfortunately Newton's ideas were far ahead of the technology of his time.

A rocket is launched by burning fuel inside of it and allowing the gases formed to escape through a small orifice at the bottom. The gases must push on the rocket in order to get out of the orifice, thus forcing the rocket upward. This is the reaction referred to in Newton's third law (Fig. 1). The force with which the rocket is propelled upward is called the thrust. The acceleration with which the rocket rises is equal to the thrust minus the weight of the rocket divided by its mass.

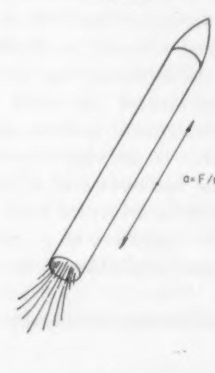
1. $a = F/m$

The problem of determining the acceleration produced in a rocket is complicated by the mass changing as the fuel is consumed. This problem requires calculus for an exact solution. However, we may assume a rocket with constant mass during flight and see the general manner in which a real rocket behaves. Let us assume a rocket of three stages with the following characteristics:

STAGE	WEIGHT (Pounds)	THRUST (Pounds)	MASS (Slugs)	TIME (Sec)
1.	15,000	44,000	468	120
2.	5,000	20,000	156	100
3.	2,000	12,000	63	80

The mass in slug units is obtained by dividing the weight by the acceleration due to gravity (32.2 ft./sec), assumed constant in the idealized situation. The time is the time of burning of the fuel, hence the time during which the thrust lasts. The

Figure 1



thrust is also assumed constant in this rocket, though it would not necessarily be so in an actual rocket. We would commit little error if we assume the thrust to be the average thrust. Friction due to the atmosphere is also ignored, though in practice it is an extremely important factor. It is so difficult to handle theoretically that friction data are usually obtained experimentally from models.*

First stage. The total weight to be propelled by this stage is 22,000 lbs. and the thrust is 44,000 lbs. giving an unbalanced force of 22,000 lbs. The mass is the total mass of 687 slugs, hence the acceleration is 22,000/687 or 32 ft./sec²—an acceleration of one g, to use technical language.

Second stage. The first stage drops at the time the second stage fires. This leaves a weight of 7000 lbs. and a mass of only 219 slugs. With an unbalanced force of 13,000 lbs. the acceleration is 59 ft./sec² or 1.83 g.

Third stage. Since the first and second stages have been dropped, a weight of only 2000 lbs. is left with a mass of 63 slugs. Since the thrust is 12,000 lbs., the acceleration is 191 ft./sec² or 5.97 g.

In uniformly accelerated motion the velocity, v, at the end of time, t, is, if the initial velocity is v_0,

$$2. \qquad v = v_0 + at$$

where a is the acceleration.

For the first stage v_0 is equal to 0, hence v is equal to 32×120 or 3840 ft./sec. At the beginning of the second stage v_0 is 3840 ft./sec, giving at the end of the second stage a velocity of 9740 ft./sec. Likewise the velocity at the end of the third stage, or final velocity, is 25,020 ft./sec, well within the orbiting range of velocities.

On the surface of the earth, for most practical problems, it is assumed that acceleration due to gravity is constant. In studying the mechanics of orbiting after the satellite is in orbit, we must take into account the variation of g. Since there are more data available about the satellites, fr . here on the calculations can be made on actual satellites. For simplicity we shall carry on with our hypothetical satellite and leave the data on actual satellites to the table at the end of this article.

Newton's law of gravitation states that: "Every object in the universe behaves as if acted upon by every other object in the universe with a force proportional to the product of their masses and inversely proportional to the square of the distance separating their centers." Hence, the force of attraction of the earth for an object will vary inversely with the square of the distance from the center of the earth. Since the acceleration given an object is directly proportional to the force, then the acceleration due to gravity at any point above the earth's surface is inversely proportional to the square of the distance from the center of the earth. If g' is the acceleration due to gravity at height, h, above the earth, then

$$3. \qquad g'/g = R^2/(R+h)^2$$

where R is the radius of the earth. This is only approximately correct since the earth is not a perfect sphere, but it is sufficiently accurate for the present purpose.

In Figure 2 the satellite is in orbit at the point P and has a velocity, v. Left to itself it would continue on a tangent to the orbit as shown by the arrow. This is in accordance with Newton's first law. Gravity, however, continually pulls it toward the earth. The force to keep the satellite in a circular path is called centripetal force and may be calculated by the formula:

$$4. \qquad F = mv^2/(R+h).$$

For the satellite to remain in orbit, the gravitational force must be equal to this centripetal force. Hence,

$$5. \qquad mg' = mv^2/(R+h).$$

Dividing both sides by m gives

$$6. \qquad g' = v^2/(R+h).$$

* All calculations slide rule accuracy only.

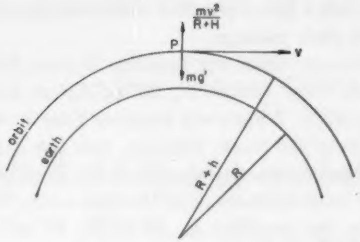


Figure 2

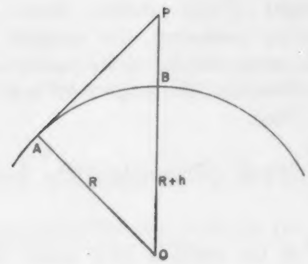


Figure 3

Solving (3) for g' and equating to (6) gives

7. $$R^2g/(R+h)^2=v^2/(R+h).$$

Therefore

8. $$h=gR^2/v^2-R.$$

If the terminal velocity of our rocket (25,020 ft./sec) and the radius of the earth in feet (21,100,000) are substituted in (8) we get the height at which our satellite will orbit, 1,800,000 ft. or 340 miles.

The reader must not get the impression from this that if the satellite reaches this velocity, it will automatically orbit. At the instant this maximum velocity is reached the satellite must be traveling on a horizontal path along a great circle. Even then it will not orbit in a circle. Due to small errors in velocity, the path being off the horizontal and the earth not being a perfect sphere, the orbit will be an ellipse. From here on we will have to speak of average velocities in the orbit, average times of revolution, etc.

The period, P, or time to make one complete revolution around the earth, may be found by calculating the circumference of the "average" circle it describes and dividing by the velocity

9. $$P=2\pi(R+h)/v.$$

Substituting in the values before obtained, this gives 5747 seconds or 95.8 minutes.

The energy of the satellite is calculated from the definitions of energy. Energy of a body is of two kinds, potential and kinetic. Potential energy in this case is the weight times the height of the satellite. The kinetic energy is one half the mass times the square of the velocity. The total energy is the sum of these two:

10. $$E=mg''h+\tfrac{1}{2}mv^2.$$

A root-mean-square average for g'' must be used. (g before was on earth. In equation 10 use g_r or g''.) The value of g' at the height h is computed from (3). The value at the surface of the earth being 32.2 ft./sec², the root-mean-square value is found to be 35 ft./sec². Let us assume that, at launching, the satellite carried in the third-stage rocket weighs 32 lbs. It will then have a mass of approximately 1 slug. The potential energy will be 45,000,000 ft.-lbs. and the kinetic energy 300,000,000 ft.-lbs., or a grand total of 345,000,000. Truly tremendous energy, but it is only a fraction of the total energy needed to put it into orbit due to the energy lost as the various stages are dropped.

It is important to know over how wide an area our satellite can be seen. If in Figure 3 our satellite is at point P, the arc represents the surface of the earth with the center at O and the satellite immediately over B, then the farthermost point at which the satellite can be seen will be A, the point of tangency of the line of sight from P. The satellite will be seen in an area enclosed by a circle on the earth with arc AB as a radius. The problem is to find arc AB in miles on the surface of the earth.

The line OA is the radius of the earth; OP is equal to the radius of the earth plus

the height of the satellite. Since, from elementary geometry, the tangent to a circle is perpendicular to the radius at the point of contact, the angle OAP is a right angle. Then

11. $\cos \angle POA = AO/OP.$

For our satellite, h is 340 miles and the radius of the earth is 4000 miles. Hence $\cos \angle POA = 0.9250$ and angle POA is 22° or 1320′. Remembering that a nautical mile is the distance of one minute of arc on the equator, arc AB is then 1320 nautical miles. To get statute miles, multiply by 6/5 or 1584 miles.

Predicting where and when the satellite may be seen is simple if we keep clearly in mind the relative motion of the satellite and earth. As the satellite revolves around the earth, the earth will rotate under the orbit of the satellite, causing the satellite to cross a given parallel of latitude farther west each passage.

Assume that our satellite is over New York City (40°48′N, 73°58′W) at 6:00 P.M. EST. Ninety-six minutes later it will be over the same latitude, but the earth will have turned under it and the longitude will be 96 minutes or 24° farther west. This puts the satellite at 40°48′N, 97°58′ or over Kansas and Nebraska. Actually the satellite will cross the 40th parallel twice in each passage. The two passages are separated by approximately 180° in longitude. The calculation of the second passage is more difficult and will not be considered here.

These calculations, approximate though they may be, should give the student an insight into the mechanics of the orbiting of man-made satellites. It is hoped they will give the imaginative student an impetus in his study of mathematics.

TABLE OF SATELLITES LAUNCHED TO OCT. 1958

Country	Weight	Max. Height	Min. Height	Period	Velocity
Russia (Oct. '57)	184 lbs.	560 miles	125 miles	96 min.	18,000 mph
Russia (Nov. '57)	1118	1020	140	104	17,800
U. S. (Jan. '58)	31	1510	218	114	19,000
U. S. (Mar. '58)	3	2458	409	134	18,365
U. S. (Mar. '58)	31	2000	100	116	18,000
Russia (May '58)	2925	1120	128	105	14,746
U. S. (July '58)	38	1200	120	112	16,000

Even in the mathematical sciences, our principal instruments to discover the truth are induction and analogy.—*Laplace*

Edited by Howard Eves, University of Maine, Orono, Maine

Michel Chasles and the forged autograph letters

by R. A. Rosenbaum, Mathematical Institute, Oxford, England

Educators often consider the question of "transfer." For example, does the study of mathematics instill habits of logical thought which carry over to the analysis of problems in other rational disciplines? It is of interest in this connection to examine the behavior of professional mathematicians, to see whether they exhibit notably logical qualities in the nonmathematical aspects of their lives. The spectacle of a mathematician acting in a markedly irrational manner fills most observers with unholy glee. No such spectacle can surpass that of Michel Chasles and the forged autograph letters.

Chasles was one of the foremost geometers of the nineteenth century. His *Aperçu historique sur l'origine et le développement des méthodes en Géométrie* · · ·, published as a memoir of the Academy of Brussels in 1837, is an extraordinary achievement of synthesis and generalization which won him immediate recognition. He contributed many theorems to geometry, and the "principle of algebraic correspondence" is known by his name [4].* Joseph Bertrand, in [1], quotes what he refers to as an oft-repeated sentence, "All the geometers of Europe are disciples of M. Chasles."

But Chasles was an especially ardent French patriot, and his nationalistic pride led to a debacle. When shown some letters, purportedly written by Pascal, in which the laws of gravitational attraction were set out, he eagerly bought them: here was proof of France's priority to Newton's England! A scholar and bibliophile with a comfortable income, Chasles continued to buy documents from one Vrain-Denis Lucas during the period 1861–69.

The details of his purchases seem incredible. He bought over 27,000 letters, for about 140,000 francs. There were 175 letters from Pascal to Newton, 139 from Pascal to Galileo, and large numbers written by Galileo [3]. But Lucas provided ancient, nonmathematical letters as well. Included in Chasles' purchases were six from Alexander the Great to Aristotle, one from Cleopatra to Caesar, one from Mary Magdalene to Lazarus, and one from Lazarus to St. Peter. Every letter was written on paper, and in French [2]! It is probably true that Chasles, in his ardor and enthusiasm, did not look at many of his 27,000 purchases.

When Chasles disclosed to the French Academy of Sciences his theory of Pascal's priority to Newton, there was considerable scepticism. Chasles displayed some of his letters, and it was pointed out that the handwriting was not the same as that of letters which were indubitably Pascal's. Various anachronisms appeared. Each was met by a new letter furnished by Lucas, in which the difficulties were explained away. But after several years of controversy, Chasles had to acknowledge defeat. He exhibited his entire stock of 27,000 forged letters, and Lucas was sent to prison for two years.

Lucas's defense at his trial was interesting. He maintained that he had done

* Numbers within the brackets refer to references at the end of the article.

nothing wrong—that Chasles had really received his money's worth, that the controversy and trial, which were widely reported, had stimulated in the public a healthy interest in history, that the debates in the Academy had been much more exciting than usual, and that he, himself, had acted through patriotic motives.

At the same time that one marvels at Chasles' gullibility, one must be amazed by Lucas's industry: to "antique" paper for 27,000 letters is itself quite a task! Lucas apparently had spent many hours each day in libraries, acquiring historical knowledge for his writing. Since he knew neither Greek nor Latin, he was severely handicapped in his work. According to Farrer [3], nothing is known of Lucas after his prison sentence; but Bertrand [1] reports that Lucas served his time, returned to his "profession" after his release, and was resentenced to three years as a recidivist, Chasles asking wryly, "Wouldn't it have been better to sentence him to five years from the start?"

Apparently Chasles was greatly embarrassed by the affair, and Bertrand remarks that the man had suffered enough—the matter should be forgotten. But Farrer can't resist giving a twist of the knife, which may well serve as a warning to all of us:

"The logical incapacity that M. Chasles displayed throughout the contest subsequently waged over his supposed treasures shows conclusively how insignificant is the benefit conferred on the reasoning faculties by mathematical studies. The leading mathematician of his country showed himself incapable of reasoning better than a child."

References

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3. J. A. Farrer, *Literary Forgeries* (Longmans, Green, and Co., 1907), Chap. xii.
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An obsolete problem in arithmetic

by Cecil B. Read, University of Wichita, Wichita, Kansas

Anyone who has browsed through textbooks of a generation ago or longer will discover that there are not only changes in the type of material which is presented, but that some problems have definitely become obsolete and have been dropped. It is interesting to note, however, the length of time it takes for a problem to disappear.

An interesting problem of this nature arose from the necessity of converting money of the various states, which eventually formed the United States of America, from one currency to another. One might, of course, expect that problems of this nature would be of importance before the forming of the nation, but it is surprising to note how long such problems remained in arithmetic textbooks after the federal government was formed.

In the sixth edition (1807) of Pike's *Arithmetick* one finds many pages devoted to problems of this nature:

To reduce New Hampshire, Vermont, Massachusetts, Rhode Island, and Virginia currency to federal money, reduce the shillings, pence, and farthings to decimals, divide the whole by

3, putting the comma [*sic*, the comma was used as we now use a decimal point] one figure to the right hand in the quotient than in the pounds of the dividend and the quotient will be the answer in dollars, cents, and mills.

To reduce New Hampshire, Vermont, Massachusetts, Rhode Island, and Virginia currency to New York and North Carolina currency, add one-third to the given sum.

. . . to Pennsylvania, New Jersey, Delaware, and Maryland currency, add one-fourth to the given sum.

. . . to South Carolina and Georgia currency, multiply the given sum by 7, divide the product by 9.

Obviously there are many variations of the problem. For example:

To reduce federal money to New England and Virginia currency; to reduce South Carolina and Georgia currency to New York and North Carolina currency (multiply the given sum by 12 and divide the product by 7).

One can see the need of many other such combinations. In addition, problems in this *Arithmetick* involved reduction of the various currencies to sterling; to Irish, Canadian, and Nova Scotian money; to Livres Tournois; and to Spanish dollars.

One might not be particularly surprised to find that problems of this nature were still of value as late as 1807. However, similar problems appear in Emerson's *North American Arithmetic*, published in 1834. Smith, in his *Arithmetic*, published in 1836, offers a double apology, once for omitting "much that is contained in other treatises respecting what is called 'the currencies of the different United States.' " He then apologizes for introducing the subject at all, saying, "Those merely nominal . . . currencies have long been obsolete in law, and ought to become so in practice. So long, however, as that practice continues, it may be necessary to retain a brief notice of it. . . . "

Even as recently as one hundred years ago, in Greenleaf's *National Arithmetic* (1858), we still find some space devoted to these problems, although the author does say, "The old currencies of the States are no longer used in keeping accounts, yet the price of articles is still named by some traders in the old currency of their State."

One might well wonder just how long it does take a problem to become completely obsolete.

An excursion into labyrinths

by Oystein Ore, Yale University, New Haven, Connecticut

Classical sources, notably Pliny, name several famous labyrinths of antiquity, probably all patterned after the original "Temple at the Beginning of the Lake," located near the town of Arsinoë in Egypt. Other types of mazes, some stone laid, some used as decorative patterns and magic symbols, may be found among many peoples. Several medieval cathedrals have floors embellished with labyrinth designs; the popularity of the garden variety of mazes goes back to the Rinascimento. Labyrinths, like most everything else, have become the object of mass production; the mazes used by the modern psychologists for their rat races exceed in number and intricacy, as well as ugliness, anything seen in earlier centuries.

Most precious to the Greeks, and particularly to the Athenians, was the labyrinth of King Minos at Knossos on Crete, the stage of the mythological love story of Ariadne and Theseus. Catullus, the Roman poet, describes the complexity of

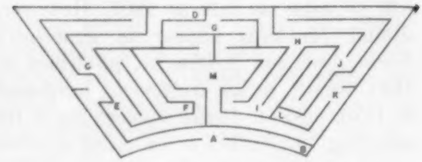


Figure 1

the labyrinth and how Theseus, after the slaying of the Minotaur, feels his way following the slender thread so ingeniously provided for him by Ariadne.

The problem of finding a way out of a labyrinth, or maze, or cave, requires systematic thought, and so should be a legitimate topic for mathematical study. Briefly, a labyrinth consists of passages leading to meeting points with other passages, and at these junctions the wanderer must make a judicious choice of new directions. Such a situation corresponds to a geometric figure called a network or a graph. A *graph* consists of *vertices*, corresponding to the junctions of the passageways, and these vertices are connected by the lines or *edges* of the graph, corresponding to the passages themselves. See Figure 1 and Figure 2, representing the maze at Hampton Court.

Mathematically our problem may then be formulated: Find a path of edges in the graph from one given vertex to another given vertex (the exit of the labyrinth). It must be assumed, naturally, that such a path is always possible, that is, that the graph has the property of being *connected*. It is also evident that it must be possible to remember which edges have been passed, because otherwise there is nothing to prevent an endless wandering around in circles. Thus we assume that the edges can be marked in some way.

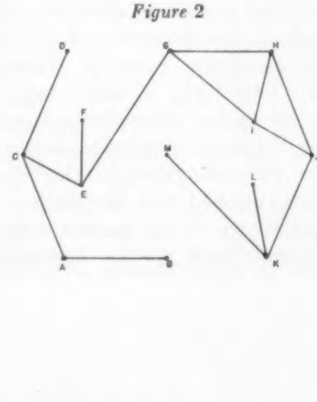
Many books on puzzles include a chapter on this labyrinth problem [1].* The first systematic procedure for finding a way out of them seems to have been proposed by Wiener [2]. His rule runs as follows: From the given vertex a_0 one proceeds along the edges of the graph as far as possible, selecting at each vertex an edge which has not previously been traversed; at a vertex where one can move no farther, the path is retraced until one arrives at some vertex where there is still some unused edge.

The last operation will consist in retracing the whole path, returning to a_0, and it is fairly clear that such a path will have covered all edges of the graph. However, it will involve many repetitious walks and the retracing of the paths requires some kind of Ariadne's thread.

In his *Recréations mathématiques* Lucas [3] analysed another procedure ascribed to Trémaux. This method results in a path in which every edge in the graph is covered once in each direction. The same result is obtained by using the following rule due to Tarry [4].

Beginning at an arbitrary vertex a_0, one follows a path P at random, marking each edge as one passes with the direction in which it has been traversed. Also, when one arrives at some vertex q for the first time, the entering edge is marked especially as such. Each time one arrives at a vertex q, one follows next an edge (q, r) which has either not previously been traversed or, if so, only in the opposite direction (r, q). If at some vertex there should be no more edges available, one makes an exit through the entering edge.

Figure 2



* Numbers within brackets refer to references at the end of the article.

One sees that by each passage through a vertex q there will be an incoming edge and a departing edge in the path P; consequently, it can only come to a halt at a_0. We shall prove that then all edges have been covered, once in each direction. First we verify this fact for all edges with an endpoint at a_0. Since P cannot be continued, all edges at a_0 must be covered in the direction from a_0; but because there are as many outgoing as incoming edges in P at a_0, each edge is covered in both directions. The same result is then obtained by induction for the other vertices in P. We assume that for vertices a_i before a vertex a_n in P the edges are covered in both directions. This, then, holds in particular for the entering edge (a_{n-1}, a_n) to a_n. Since the entering edge has been used for an exit, there can be no other edges at a_n which have not been covered in the direction from a_n. But again there are as many outgoing as incoming edges at a_n in P, and so all edges are covered in both directions.

The Tarry paths show that it is always possible to pass through a connected finite graph in such a manner that each edge is covered once in each direction. This means, for instance, that one can walk through an exhibition by such a route that the exhibits on each side are all inspected without repetition.

The labyrinth methods mentioned are ingenious, yet it seems that none of them is quite appropriate for the problem at hand. If a wanderer is lost at some point a_0, he must have arrived there by some bounded walk, and the exit must lie at a not too distant point. But then it should be unnecessary to meander through the whole labyrinth to its most distant recesses. What is wanted instead is a method of search which insures that all vertices within a certain range have been visited.

In a graph one usually introduces a *distance* $d(a, b)$ between two vertices a and b as the smallest number of edges in any path between them. Thus two vertices connected by an edge have the distance 1. To visit all vertices at the distance 1 from a_0 is easily done: one passes along each edge at a_0 to its other end and returns to a_0. But in order to continue systematically it is necessary to mark the edges in some way. Each edge $E=(a_0, a_1)$ is marked once as one leaves a_0 and at the point a_1 it is marked as the entering edge. If there should be no edges at a_1 other than E we return to a_0 and mark E as closed. If some other edge $E=(a_0, a_1)$ should also lead to a_1 from a_0, we mark it closed at both ends; the same is done with any circular edge returning to a_0.

To visit all vertices at the distance 2, one selects some unclosed edge $(a_0, a_1)=E$ at a_0 and marks it again. At a_1 the nonclosed edges are traversed and marked, eventually closed if they reach vertices already touched upon. When this is completed one returns to a_0 by the entering edge. If there should be no edges open at a_1 after the visit, the entering edge E is also marked closed at a_0. After the return to a_0, the same process is repeated on one of the other open edges marked only once, and this is repeated until they all have been treated.

After having touched upon all vertices at a distance at most n from a_0, the situation is as follows. All open edges at a_0 are marked n times, the open edges at a vertex a_1 of distance 1 are marked $n-1$ times, and so on. To visit next the vertices at a distance $n+1$, one moves successively to each vertex a_1 at a distance 1 and visits all vertices at a distance n from a_1, using only the edges still open and marking and closing edges according to the rules given above.

The description of the method may seem a little complicated. In actual examples the procedure simplifies a good deal because so many edges soon become closed. The reader may try it out on the famous maze from the garden at Hampton Court.

The methods of Trémaux and Tarry also had a theoretical implication for graph theory because they show that any finite

connected graph can be traversed in a path which includes each edge just once in each direction. The last method, varied slightly, also yields a fact of theoretical interest. We modify the preceding rules so that an edge is eliminated or closed only in case it goes to a vertex reached previously. It is seen that the remaining edges in the graph form a *tree*, that is, a connected graph T without circular paths. One easily verifies that this tree has the following properties: There is at least one edge of T at each vertex in the graph G, and T is a maximal tree. That is, when any edge of G is added to T it will produce circular paths.

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2. Wiener, C.: "Über eine Ausgabe aus der Geometria situs," *Math. Ann.*, VI (1873), 29–30.
3. Lucas, E.: *Recréations mathématiques*, four volumes (1882–1894).
4. Tarry, G.: "Le problème des labyrinthes," *Nouv. Annales de Math.*, 3d. ser., XIV (1895), 187–90.

What's new?

BOOKLETS

Algebra Can Be Fun, William R. Ransom. Portland, Maine: J. Weston Walch, Publisher, 1958. Paper, ix+195 pp., $2.50.

Calculus Quickly, William R. Ransom. Portland, Maine: J. Weston Walch, Publisher, 1958. Paper, viii+60 pp., $1.00.

Introduction to Algebra and Indirect Measurement, Louise A. Mayer. Benton Harbor, Michigan: Educational Service, Inc., 1957. Paper, vi+110 pp.

Money Management, Your Savings and Investment Dollar. Money Management Institute, Household Finance Corporation, Prudential Plaza, Chicago 1, Illinois, 1959. Paper, 40 pp., 10¢.

Rapid Analytics, William R. Ransom. Portland, Maine: J. Weston Walch, Publisher, 1958. Paper, vii+55 pp., $1.00.

INSTRUCTIONAL MATERIALS

Data-Guide: Trigonometry. Data-Guide Distributing Corp., 4005 149th Pl., Flushing 54, New York. 8½×11 plastic sheet written by Arnold Fass giving summary of trigonometry, 79¢.

Basic Kit of Mathematics. Harvey House Publishers, Irvington-on-Hudson, New York. Kit consisting of a book, *The Story of Mathematics* by Hy Ruchlis and Jack Engelhardt, three-dimensional geometric forms patterns, compass, protractor, triangles, ruler, and color pencils, $2.95.

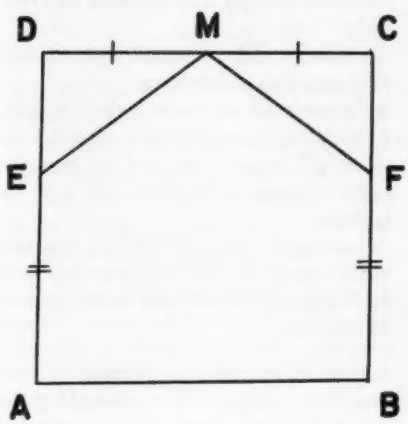
The most simple relations are the most common, and this is the foundation upon which induction rests.—*Laplace*

Edited by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

Proofs with a new format

by Emil Berger, Monroe High School, St. Paul, Minnesota

Modern curriculum proposals are focusing attention on logic and mathematical structures. The proofs of geometry have usually been the models of a mathematics deduction. Frequently, however, the logic of the deduction is hidden in the repetition of the steps of formal proofs. I have tried a new format in my geometry classes and find that it makes more sense to my students than the typical textbook proof. The sample below is an actual illustration of a student's paper which uses the "new" format:



Given: Square $ABCD$ with M the midpoint of DC and $AE=BF$.

To prove: $ME=MF$.

Proof:

1. *Assumption:* Perpendicular lines form right angles which are equal.
2. *Given:* $DC \perp CB$ and $CD \perp DA$.
3. *Deduction:* Angles C and D are equal right angles.
4. *Assumption:* The midpoint of a line bisects it into two equal segments.
5. *Given:* DC with midpoint M.
6. *Deduction:* $DM=CM$.
7. *Assumption:* If equals are subtracted from equals, the differences are equal.
8. *Given:* $AD=CB$; $AE=BF$.
9. *Deduction:* $CF=DE$.
10. *Assumption:* S.A.S.=S.A.S.
11. *Given:* $DM=CM$; $\angle C=\angle D$: $CF=DE$.
12. *Deduction:* $\triangle DEM \cong \triangle CFM$.
13. *Assumption:* Corresponding parts of congruent triangles are equal.
14. *Given:* $\triangle DEM \cong \triangle CFM$; $DM=CM$; $\angle C=\angle D$; $CF=DE$; ME corresponds to MF.
15. *Deduction:* $ME=MF$.

Similar polygons and a puzzle

by Don Wallin, Riverside-Brookfield High School, Riverside, Illinois

The combination of a series of polygons can be a unique basis for applying various computational and constructional procedures during a unit on similar polygons. The polygons used are cut out of cardboard as shown in Figures 1–5 and have these dimensions:

1. An isosceles right triangle with legs two inches long.
2. An isoceles right triangle with an hypotenuse two inches long.
3. A square with sides one inch in length.
4. A pentagon composed of a square with sides $\sqrt{2}$ inches long and an isosceles right triangle with legs one inch in length.
5. A pentagon composed of two isosceles right triangles, one with legs one inch in length and the second with legs two inches long.

Additional polygons are formed by combining these polygons. For example, polygon 6 is a square formed by combining polygons 1, 2, 4, and 5. It is a challenging puzzie to figure out how to form this square with the given polygons.

Another puzzle requires the use of all five polygons to form polygon 7. This polygon is also a square. To help students form this square it is usually necessary to tell them to compute the area of the square. Knowing the length of the side of the square aids in placing the polygons in proper position.

The following problems are given as exercises in comparison of lengths and areas of the various similar polygons. For simplicity, the ratio of any two corresponding sides, altitudes, and perimeters of two similar figures is called the "linear ratio."

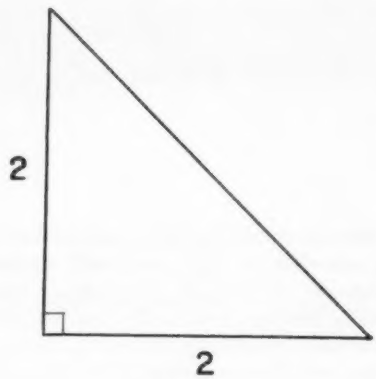


Figure 1

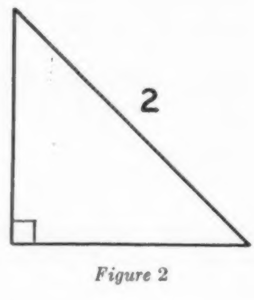


Figure 2

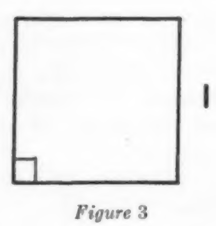


Figure 3

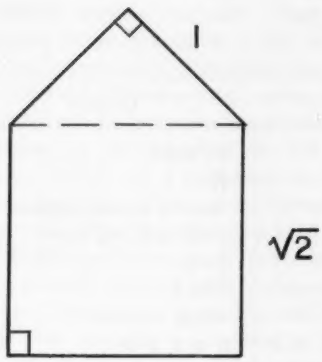


Figure 4

1. Why are polygons 1 and 2 similar?
2. What is the linear ratio of 1 to 3?
3. What is the ratio of the areas of 1 to 2?
4. What are the missing dimensions for each polygon?
5. Why are polygons 3, 6, and 7 similar?
6. What is the linear ratio of 3 to 6? 6 to 7?
7. What is the ratio of the areas of 3 to 7? 6 to 7?
8. What is the combined area of 3 and 4?
9. What is the ratio of the areas of 3 and 6 to 7?
10. What is the ratio of the area of 1 and 2 to 5?

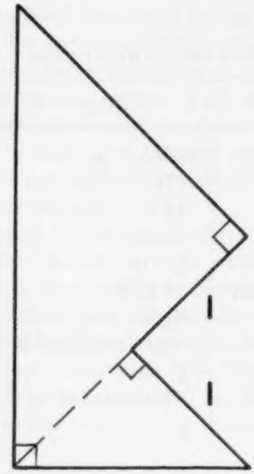


Figure 5

Finally the following constructions are completed:

1. A square with the area of polygon 6.
2. A square with an area equal to the sum of the areas of polygons 3 and 6.
3. A triangle equal in area to the sum of the areas of polygons 1 and 2.

This set of puzzles, problems, and constructions has added variety to the usual textbook problems on similar polygons.

Mathematics rummy

by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

This game is played like animal rummy. However, instead of animal pictures the cards have number facts or problems. The object of the game is to get rid of cards by forming "books" or "sets" of cards. A "book" is a set of 3 or 4 cards of the same number fact, as $\frac{3}{4}$, .75, 75%, $\frac{12}{16}$, or problems with the same answer, 3×4, 4×3, $4+4+4$, $3+3+3+3$. A "set" is 4 or more cards with the same number value and/or process, as 3×4, 2×6, 50% of 24, area of a triangle with altitude 6 in. and base 4 in. The deck should contain from 40 to 60 cards and may be played by 2 to 6 players.

To play, shuffle the cards and deal one at a time face down until each player has

TABLE 1

Sample number rummy cards

Red	*Blue*	*Green*	*Orange*
4×3	3×4	$3+3+3+3$	$4+4+4$
$\frac{3}{4}$	.75	75%	$\frac{15}{20}$
9×4	12×3	50% of 72	Area of 6-inch square
6 ft.	72 in.	2 yd.	1 yd. 2 ft. 12 in.
$\frac{2}{3}$ of $\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}\times\frac{2}{3}$	$\frac{3}{4}\div\frac{3}{2}$
60% of 75	$.6\times75$	$\frac{6}{10}\times75$	$75\times\frac{3}{5}$
integer	digit	5	whole number
$3x^5$	The coefficient is 3	The exponent is 5	$3\cdot x\cdot x\cdot x\cdot x\cdot x$
125%	$\frac{5}{4}$	1.25	$1\frac{1}{4}$
$35\div5=7$	The quotient is 7	The dividend is 5	If 5 is subtracted from 35 seven times the remainder is zero
$3\times4\times5$	$5\times3\times4$	3×20	4×15
$p=br$	$p\div b=r$	$p\div r=b$	The product of the base and rate gives the percentage
3^2	9	$\sqrt{81}$	Area of a square 3 units on a side

TABLE 2

Sample algebra rummy cards

Red	*Blue*	*Green*	*Orange*
x^3	$x\cdot x\cdot x$	volume of cube	exponent is 3
$3x$	$x+x+x$	perimeter of equilateral triangle	coefficient is 3
3	odd number	integer	digit
$\sqrt{x}$	$x^{1/2}$	square root of x	one of 2 equal factors of x
$x/y=3$	$x=3y$	y is a factor of x	equation with two variables
π	3.14	ratio of circumference to diameter	$\frac{22}{7}$
x^2	$x\cdot x$	area of a square x units on a side	the square of x
$x>3$	$2x>6$	solution is set of numbers greater than 3	inequality
$x=7$	$x+5=12$	the root is 7	linear equation of one place holder
$x^2=9$	$x=3$	$x=-3$	quadratic equation
60%	.60	$\frac{3}{5}$	per cent
x	general number	place holder in an equation	variable

6 cards (7 cards if two play). Place the remaining cards face down in the center of the table to form a "drawing pile." Turn the top card of the drawing pile face up beside the pile to start the discard pile. Play is begun by the player at the left of the dealer, who draws a card from the top of either the drawing pile or the discard pile; if possible, this player forms a book or set, which he lays face upward before him. He then discards one card to the discard pile. He may form more than one book or set if he can, but may only draw or discard one. This order of draw, play, and discard is followed by each player in turn.

Any player, in turn, may play the 1 remaining card to a book turned up on the table. To any set of cards may be added any card or cards of the same value or process by any player. If all cards in the drawing pile have been used before the game is won, the discard pile may be shuffled and turned face down to form a new drawing pile.

A game is completed when one player lays down all his cards or when no cards remain in the drawing or discard piles. All players then total the number of cards in their hands. The player with the smallest number of points wins. The mathematical expressions to be matched can be varied to apply to the topics being studied.

. . . Until about 1951 upper-grade enrollment may be spoken of as a highly select group preparing for higher education. . . . Only after 1952 for the first time in history of Russian education did upper-grade enrollment surpass its 10 per cent share in total enrollment.

. . . While it is misleading to compare American high-school enrollment, it must be stressed again that the enrollment trends stressed above support the contention made earlier that in the past only a small percentage of Soviet students who enter the first grade are eventually graduated from the tenth grade. In the U.S. about 55 per cent. . . . In the Soviet Union only about 5 per cent in the past, and recently about 12 per cent, of those who enter the first grade graduate ten years later.—Soviet Professional Manpower, *National Science Foundation*, 1955, p. 61–62.

• POINTS AND VIEWPOINTS

A column of unofficial comment

Thanks a million

by H. Van Engen, University of Wisconsin, Madison, Wisconsin

With this issue, the present Editor's responsibility for THE MATHEMATICS TEACHER comes to an end. At the Christmas meetings in New York, the Board of the National Council of Teachers of Mathematics appointed Dr. R. E. Pingry of the University of Illinois to the office of Editor of THE MATHEMATICS TEACHER. Dr. Pingry will be responsible for all future issues.

The Council is indeed fortunate to have Dr. Pingry as editor of the magazine. He is well known for his work in mathematics education and for his work with the Council. The magazine is in good hands for the next three years.

In bringing my editorship to a close, it is indeed pleasant to acknowledge my indebtedness to many people for their contribution to the editing chores. It is unfortunate that all who assisted cannot be named. Literally hundreds of people referred manuscripts, wrote reviews, commented on special policy questions, and assisted in a variety of ways. The magazine would have suffered greatly without their help.

To Dr. Irvin Brune, assistant editor for the past six years, the Editor wishes to express his sincere appreciation for his valuable help. Dr. Brune prepared the index for each volume of the magazine, read manuscripts, and gave good advice. The magazine would have suffered without his valuable help.

A special vote of thanks must go to department editors, both past and present. Throughout the past six years a number of people assisted by taking over responsibility for a section of the magazine. Records show that, at one time or another, the following people served as department editors:

Emil J. Berger
John A. Brown
Kenneth E. Brown
Paul C. Clifford
Richard D. Crumley
Dan T. Dawson
Howard Eves
Robert S. Fouch
Donovan A. Johnson
Phillip S. Jones
Robert Kalin
Houston T. Karnes
Lucien B. Kinney
Francis G. Lankford, Jr.
William C. Lowry
Roderick C. McLennan
Sheldon S. Myers
Joseph N. Payne
Philip Peak
Cecil B. Read
William L. Schaaf
Adrian Struyk
Henry Syer

The Editorial Board has always given the Editor splendid support and advice. Their support is appreciated and also their advice even though the Editor was not always able to follow the Editorial Board's excellent counsel. The following served as members of the Editorial Board:

1953–1956

Jackson B. Adkins
Phillip S. Jones
Z. L. Loflin
Philip Peak
M. F. Rosskopf
Helen Schneider

1956–1959

Jackson B. Adkins
Mildred Keiffer
Z. L. Loflin
Philip Peak
Ernest Ranucci
M. F. Rosskopf

Thanks, also, to these people for their help.

The Board of the National Council of Teachers of Mathematics has been very helpful by listening sympathetically to all editorial troubles and woes. Many members of the Board have been helpful in various ways.

M. H. Ahrendt, executive secretary of the National Council of Teachers of Mathematics, has always been responsible for the advertising which has appeared in the magazine. For this and his assistance in many other ways, the Editor is indebted to him.

And last of all, all the members of the Council deserve a hand for the "Letters To the Editor"—you should write more—and for reading the magazine. Where would a magazine be if it did not have readers?

Nineteenth Summer Meeting

University of Michigan

August 17, 18, 19, 1959

The Nineteenth Summer Meeting of the National Council of Teachers of Mathematics will be held at the University of Michigan, August 17–19.

Special features of the program for this meeting include experimentation and experiences in teaching new topics in mathematics classes in the elementary school, junior high school, and senior high school; report from the School Mathematics Study Group who will be completing their second summer of writing by this time; the Arithmetic Project at the University of Illinois; programs for both gifted and non-gifted students of mathematics, including a demonstration lesson in an elementary school class; mathematics education in other countries; foundations in mathematics; responsibilities and opportunities for the mathematics consultant; a series of four lectures in mathematics; and pre-service and in-service programs for mathematics teachers.

We believe the program will bring inspiration and many practical ideas to those who attend. We hope you plan to come.

NCTM-NEA Summer Meeting

The NCTM plans two excellent programs as parts of the NEA Convention in St. Louis. Both meetings are July 1, 1959, in the Melbourne Hotel. At 9:30 A.M. a panel, principally administrators, will give ideas and plans for taking care of the pupil superior in mathematics. This will be followed by questions and discussions from the floor. The luncheon speaker will give an overview of status and goals of the various groups now working to improve curricula and methods in mathematics.

Jesse Osborn

Reviews and evaluations

Edited by Richard D. Crumley, Iowa State Teachers College, Cedar Falls, Iowa

BOOKS

Editorial Note: For books of great importance to the teaching profession, THE MATHEMATICS TEACHER has a policy of providing dual reviews written by people who may look at the book from different viewpoints.

Teaching Arithmetic for Understanding (and student handbook), John L. Marks, C. Richard Purdy, and Lucien B. Kinney (New York: McGraw-Hill Book Company, Inc., 1958). Cloth, xiv+429 pp., $6.00.

Many of the criticisms stated here apply equally to most other books on the teaching of arithmetic. On some of these points the present book is better than most.

I find no clear statement of the nature of mathematics as the science of deductive reasoning, nor of mathematics as an art and as a language, nor of the role of mathematics in the natural and social sciences. I find little mention of mathematicians, nor any hint that mathematics is created by human beings. A reference to Bell, *Men of Mathematics*, and to some book on the history of mathematics, would help. Every teacher should know Kline's book, *Mathematics in Western Culture*. There is no adequate reference to career information and the needs for mathematical training in various vocations, nor to the manpower needs of our society, such as the Steelman report, *Science and Public Policy*. The outline of the aims of arithmetic study on page 12–14 is unsystematic and confusing, and is hardly a model of clear thinking for the prospective teacher. There is no mention of the impact of high-speed computers on our society. There is no mention of elementary number theory in either the text or the teachers' manual. Such a book as Ore's *Number Theory and its History* is an invaluable source of enrichment material. How can a teacher or a teacher of teachers make arithmetic interesting if he doesn't know anything interesting about numbers? There is no reference to Swain, *Understanding Arithmetic*, which is the best book I know on the mathematical content of arithmetic.

The authors do not mention the fundamental work of Piaget on the development of mathematical concepts in children. A teacher would find helpful a summary of the research on learning and reinforcement and its application to the scheduling of drill.

The authors recommend, essentially, the grade placement of topics suggested by the Committee of Seven. It was thirty years ago when Brownell published a devastating analysis of this research. Yet the results of the Committee's research are used as though they provide reliable evidence for the design of the curriculum. In England, Sweden, and other countries, children learn in four years what our pupils learn in six. The report of Buswell indicates that British children do learn this curriculum effectively. The concept of "social uses" of arithmetic is narrowly limited to routine computations and ignores the applications to the natural social sciences.

There is a good discussion of the meaning of the decimal system of notation and alternate bases. It is not made clear that numerals are names of numbers, and that the content of arithmetic is the properties of numbers. The notation must be understood, but is incidental, as a tool of communication, to the main business of teaching arithmetic.

The authors lay a welcome stress on the laws of arithmetic, and discuss explicitly the commutative and distributive laws. The associative law appears disguised as the law of "compensation." These generalizations are well stated in words, but the authors apparently feel that the statements in symbols are too difficult not only for teachers, but also for teachers of teachers. Average sixth graders learn with delight that if a, b, and c are any numbers, then

$$a \cdot (b+c) = (a \cdot b) + (a \cdot c),$$

and that this is the explanation of the usual process for computing

$$3 \times 21 = 3 \times (20+1) = (3 \times 20) + (3 \times 1).$$

Like all educators, the authors advocate adjustment for individual differences, yet do not, apparently, apply this philosophy in their own teaching. There is nothing in the text adapted to the capacities of a student who has had ninth-grade algebra, let alone any college mathematics. I can think of no policy more likely to discourage a well-prepared student from going into elementary teaching.—*Paul C. Rosenbloom, University of Minnesota, Minneapolis, Minnesota.*

Teaching Arithmetic for Understanding (with teacher's manual and student workbook), John L. Marks, C. Richard Purdy, and Lucien B. Kinney, New York, McGraw-Hill Book Company, Inc., 1958. Cloth, xiv+429 pp., $6.00.

Research by Grossnickle and others has revealed the lack of substantial mathematical background and good procedures for the teaching of today's arithmetic. Here is a publication which attempts to meet the challenge by pro-

fessionalizing the subject matter for pre-service and in-service teachers (grades 1–8). The authors maintain theirs is a dual purpose when they say: "So that the teacher may thoroughly comprehend the mathematical aspects of elementary school arithmetic, mathematical ideas such as properties of our number system, why numbers are placed in certain positions for computation, and the principles and relations underlying the operations are stressed"; and, "Above all else the emphasis here is upon strategy and techniques for use in helping pupils develop understanding of arithmetic." It is the opinion of this reviewer that the authors, on the whole, have done remarkably well in dealing simultaneously with mathematical substance and methods. Indeed, for the professionals who know Buckingham's *Elementary Arithmetic: Its Meaning and Practice* and the many articles written by Brownell and Van Engen relative to the philosophy and psychology of arithmetic, this publication may appear to be a good integration of all three.

The scope of the text is adequate. Its fourteen chapters include: (1) Why Study Arithmetic; (2) Planning Effective Learning Activities; (3) The Arithmetic Curriculum; (4) Number and Number Systems; (5) Beginning Number Experiences; (6) Addition and Substraction; (7) Multiplication and Division; (8) Common Fractions; (9) Decimal Fractions; (10) Learning the Language of Per Cent; (11) Measures and Measuring; (12) Learning to Solve Word Problems; (13) Appraising Progress of Pupils; (14) Adjusting to Individuals. Also included is a section on "Games for Fixing Skills" and one on "Selected References." There is a name index as well as a subject index. Within each chapter, the sequence and the techniques for developing number ideas are commendable, as are materials and devices.

Mathematically the content is weighted heavily in favor of the collectional approach to number. It would seem that addition and subtraction could be strengthened and zero made more significant had the number line been given greater prominence. In our time, thinking with positive *and* negative numbers has become almost commonplace in the elementary school. Likewise, more emphasis on equational thinking in items such as $n+5=12$ and $9-n=6$ as well as $\square+\square=7$ and $\triangle+\triangle=1$ would invigorate mathematical insight.

Sensible algorisms are employed in the discussion relating to common fractions, and it was good to see "$\frac{1}{2}+\frac{1}{3}=?$" in the horizontal form as well as the vertical. Division of fractions is presented adequately for practical purposes, but the rationale for the inverted divisor will be found in a much later section called "Experiences for Enrichment." The rationalization of multiplication and division of decimal fractions gets very extensive treatment, and at times the use of the words "rule," "hypothesis," "theory," "law," and "generalization" in this section may be confusing.

Occasionally the authors lapse into the older pattern of verbalizing mathematical ideas, and sometimes it is acknowledged as such. It would seem that "borrow," "cancel," "reduce," and "to the right of the decimal point" have had their day and that in the interest of good mathematics these should give way to "change," "simplify," and "to the right of the ones' place."

Following each chapter there are numerous stimulating questions and exercises succeeded by an extensive supporting bibliography. Missing, however, is reference to Brownell's "Psychological Considerations in the Learning and Teaching of Arithmetic" (*Tenth Yearbook*, NCTM, 1935) and Van Engen and Gibb's *Mental Functions Associated with Division* (Iowa State Teachers College, 1956). While McConnell's "Recent Trends in the Learning Process" (*Sixteenth Yearbook*, NCTM, 1941) is included in the bibliography for Chapter 2, none of the preceding twenty-five questions or exercises refer to it. Arithmeticians consider this a most significant investigation of generalization in the field of number. Indeed, it is one of the first important studies relative to drill vs. understanding and, in the opinion of this reviewer, it deserved a focal question or exercising.

Though there is a workbook to provide further exercises as well as study questions, according to the authors, its assignments can be made for use with other arithmetic methods textbooks. The *Teacher's Manual* supplies answers to the text and workbook questions and it provides examples of test questions (without answers).

These materials by Marks, Purdy, and Kinney make an ambitious and concentrated program intended for one semester of forty-five or fifty meetings. Thus selectivity will be pertinent to insure the best use of their many suggested activities. And, while the text will assist some college instructors, it cannot be considered the only answer to a good mathematical background for elementary school teachers.—*Ann C. Peters, State Teachers College, Keene, New Hampshire.*

Arithmetic in My World (grade 7), C. Newton Stokes, Paul J. Whiteley, and Humphrey C. Jackson, New York, Allyn and Bacon, Inc., 1958. Cloth, 384 pp., $3.04.

Arithmetic in My World (grade 8), with teachers' edition, C. Newton Stokes, Paul J. Whiteley, and Anne Beattie, New York, Allyn and Bacon, Inc., 1958. Cloth, 383 pp., $3.04.

In the teacher's edition of *Arithmetic in My World*, the authors make the following statements concerning their theory of teaching arithmetic. "The modern teaching program must have a social approach; attention must be given to adjustments in living. . . . the study materials for this program should be based on

reality: the problems of the child's own needs and interests."

To get information concerning the child's needs and interests, the authors, in a recent ten-year period, with the co-operation of parents and teachers of some 72,000 children in grades 1 through 8, collected data on the everyday problems of children. The data was analyzed in light of the arithmetic which would be needed for solution of these problems. From this analysis, the authors felt that they could decide what problems should be studied in the classroom and what arithmetical concepts should be acquired at different age-levels to insure well-adjusted living. A problem was considered worthy of inclusion if it was encountered by 60 per cent of the children studied at a particular grade-level. As a result, the seventh- and eighth-grade books in the series devote considerable space to work with commission, discount, mail orders, meters, calipers, registered mail, banking, installment buying, insurance, borrowing, stocks and bonds, and other problems from the physical world. The arithmetical concepts presented are examined in terms of their social applications.

The books are attractively and sturdily bound, and color has been generously used to make the pages inviting. The arithmetic, arranged as it is around social situations, may appeal to students. The mathematics presented is, in general, sound, although the reviewer feels it necessary to take issue with the authors on some points (e.g., reference to a ratio as a fraction, and the suggestion that pupils *prove* pi equals $3\frac{1}{7}$ by computing the ratio of circumference to diameter using pupil measures of one circular object).

The reviewer has some reservations concerning the validity of conclusions drawn from the survey mentioned above. It is difficult to accept that the quantitative social situations to be studied at the seventh- and eighth-grade levels were very real problems to the twelve- and thirteen-year-old pupils studied. A question might also be raised concerning whether acquiring knowledge about these situations will cause any significant change in the life-adjustment of a child in early adolescence.

Whether *Arithmetic in My World* is selected as a textbook by an arithmetic teacher will depend upon the teacher's own theory of teaching arithmetic. If the teacher feels as the authors feel, that arithmetic must have a social approach, he will find these books a well-organized source of materials for his program. If, however, the teacher believes that the mathematics is the core around which the program must be organized and that social situations are important only as applications of the mathematics, he will find the books unsatisfactory. Apart from the unique approach to determining content, these books are traditional junior high school arithmetic books. The teacher interested in the modern approach to teaching mathematics must look elsewhere for his materials.—*Della L. McMahon, Iowa State Teachers College, Cedar Falls, Iowa.*

Introducing Mathematics (2nd ed.), Floyd F. Helton, New York, John Wiley and Sons, Inc., 1958. Cloth, xv+396 pp., $5.75.

This text attempts to present, on a mature level, mathematics for freshman college students with little or no background in high school mathematics. It covers, but with relatively little depth, the material of junior high school mathematics, elementary high school algebra, and elementary high school geometry. Thus the content is quite traditional. Likewise, the presentation of the material is quite traditional.

There is good emphasis on the relations between subtraction and addition, and between division and multiplication, thus utilizing well the notion of inverse operations. Another good point of the text is that systems of simple linear equations are introduced early in the section devoted to algebra.

There are some spots and areas that to this reviewer seem to represent weaknesses. For example, no negative numbers are used until the middle of the book. This seems to be quite a handicap, for example, in the section dealing with scientific notation. There, scientific notation symbolism cannot be used for decimal fractions. When the laws for positive integral exponents are explained, the author makes the comment that they are not true in case the base is equal to zero. This comment is not true. In the rather extensive section devoted to manipulation and simplification of algebraic fractions, there are no precautions given about the possibility of division by zero in the simplification of these fractions. However, earlier in the textbook the question of division involving zero is treated very well. Unfortunately the definition of $\sqrt{x}$, on page 243, includes the possibility of its being a negative number. Also no restrictions are placed on the number x, with the result that $(\sqrt{x})^2=x$ appears to be true for all x. The damage is repaired slightly later on in an example to show that principal square roots are always positive. Finally, on page 244 is a statement that the proofs of the laws of radicals depend upon fractional exponents. However, the laws of radicals, as defined by this author, can be proved from these definitions.

In summary, this is a book written for the college level, containing high school mathematical content, pitched at a junior high school level of understanding. It is unfortunate that many colleges throughout the United States find it necessary to offer courses based on textbooks like this for college credit.—*Lyman C. Peck, Ohio Wesleyan University, Delaware, Ohio.*

Introduction to Logic and Sets (preliminary ed.), Robert R. Christian, Boston, Ginn and Company, 1958. Paper, vi+70 pp., $0.90.

An Introduction to Sets and the Structure of Algebra, W. R. Krickenberger and Helen R. Pearson, Boston, Ginn and Company, 1958. Paper, iv+32 pp., $0.60.

One should view both booklets as welcome additions to the mathematics teacher's library.

Although the subjects of logic and sets are still, unfortunately, quite strange to many teachers of mathematics, it is now more likely than ever before that they will seek the kind of material represented in these booklets.

The pace of development in both booklets is quite brisk, probably too brisk for the teacher and student whose acquaintance with the subject is still in the embryonic stage. The study of these booklets should encourage one to search for further materials on the same and related topics.

The first half of the first booklet is devoted to the subject of logic, the second half to sets. The author uses well in Part II much of the content developed in Part I, thus demonstrating that the knowledge of some logic is quite desirable when studying sets.

A well-informed teacher should make good use of the ample supply of problems, supplementing them with additional practice problems of his own.

One cannot help agreeing with the author's claim that "the *language* of logic and sets must sooner or later play an important part in elementary mathematical instruction." Considering the recent expansion of much effort to "modernize" the mathematics curriculum, one should view with optimism the possibility that the language of logic and sets will be soon a part of repertory of every mathematics teacher. Both booklets contribute well toward this end.

The second booklet, smaller in scope, seems to cover much in little space. It is only fitting that a reviewer should find some faults. One must hasten to add, however, that a minimum of effort could produce much improvement.

The authors of the second booklet failed to make clear distinction between members of sets and names of members. For example, on page 1, the sentence "... we designate a set by enumerating, or listing, its members within braces" is ambiguous. What the authors, no doubt, mean is that one refers to a set by writing *names* of its members within braces. This difficulty recurs in several places throughout the booklet.

The notion of identical sets is also unclear: "Two sets are said to be identical when they have identical members" (p. 3). It should be obvious that authors do not intend to deal with *two* sets when they speak of identical sets. There is only *one* set involved here. In connection with identical sets, the authors speak of equality. They claim that, in this context, the symbol "=" does not mean "equal" in "the usual sense." One must accuse most of the mathematics textbooks of the failure to make clear what "the usual sense" of the symbol "=" is. Perhaps it is proper to suggest the following definition:

$$a = b$$

means

"a" and "b" are two names for the same thing.

The sentence "Then the subsets of all rational numbers, of all real numbers, and of all complex numbers are fields" is totally incomprehensible. The authors probably intended to state that a subset of the universe S consisting of all rational numbers is a field. The concept of a field should be given a more thorough treatment.

The use of adjectives as names for members of sets is a further example of this difficulty. Such use is inappropriate.

The second booklet contains bibliography of seven well-chosen titles which should prove of benefit to teachers of mathematics. One could add several other sources for reference. Surely, the forthcoming *Twenty-fourth Yearbook* of the National Council of Teachers of Mathematics will merit listing in this booklet.

One can only hope that more booklets concerned with the content new to secondary mathematics will make their appearance in the near future. The two booklets reviewed here are an encouraging beginning.—*Eugene D. Nichols, Florida State University, Tallahassee, Florida.*

The New Mathematics, Irving Adler, New York, The John Day Company, 1958. Cloth, 187 pp., $3.75.

During the past few years there has been an increasing emphasis on mathematics as a study of structures. The secondary teacher has found it necessary to become well acquainted with those basic structures encountered in elementary mathematics: group, ring, and field. For the teacher who has had an introductory course in abstract algebra this poses no particular problem. For others the search for information to clarify these concepts has, until recently, been made unnecessarily difficult by the absence of suitable materials. Although it is possible to ferret the required concepts out of an algebra text, the price, in terms of time and effort, is much higher than many a teacher can afford to pay. Professional articles, National Science Foundation Institutes, and in-service programs have attempted to help bridge the gap. Now here is a small book written in clear, concise language that makes these ideas readily available to both secondary teachers and their students.

The New Mathematics is an introduction to some of the fundamentals of modern algebra for the nonspecialist. It is not by intent a text, although with additional problem material it might very well serve as one. This book traces the development of the number system from the ordinary whole numbers used for counting through the complex numbers. En route the reader becomes acquainted with groups, rings, fields, vector spaces, and topological spaces. These concepts are carefully and clearly presented. The technical language is kept to a minimum. Examples from the reader's past experience are used to fix the new concepts. "Do It Yourself" problems appear at the end of each chapter. There is also a helpful summary of basic definitions, and an index.

The New Mathematics has a place on the bookshelf of every secondary teacher. It can be used by the interested high school student who wishes to know more about the mathematics he studies as well as by his instructor who has the same goal in mind.—*Augusta Schurrer, Iowa State Teachers College, Cedar Falls, Iowa.*

The Structure of Arithmetic and Algebra, May Hickey Maria (New York: John Wiley and Sons, Inc., 1958). Cloth, xiv+294 pp., $5.90.

"This book is an elementary axiomatic development of the real number system. Its aim is to make available . . . to the teacher of secondary school mathematics the fundamental concepts that underlie the structure of algebra and arithmetic."—Preface. It begins with postulates such as the commutative laws, which are in some ways more convenient starting places than the Peano postulates. The properties of zero and of additive inverses are *derived* in Chapter 3 in a way reminiscent of the *construction* of negative numbers in some texts on modern algebra. Mathematical induction and the natural numbers are not reached until Chapter 8—an indication of the slow thoroughness of the development. This is an excellent book for readers who wish an extremely detailed but elementary treatment. Its only motivating devices are "internal." (For example, the need for a new axiom "to identify in the real number system other elements that are non-rational" is discussed at length.) The book neglects, however, the "external" motivation of axiomatics which comes from introducing a wide variety of number systems, finite and infinite, commutative and otherwise.

Theorems about the real numbers in their full generality, including irrationals, are proved in chapter 12, from a continuity axiom on the existence of least upper bounds. Limits of sequences are discussed, and decimal expansions two chapters later; giving a substantial overlap with "advanced calculus" texts.

One gentle criticism: This reviewer believes that as teachers of mathematics we would do well to employ language with more significance and less bleak austerity than that now current. Consider, for example: "Here mathematics meets the problem of how to talk about its abstract entities, of what to say about them as mere blanks. What mathematics does in fact is to study not the entities themselves but relationships between a system of entities" (page 17). Instead of speaking of the relationships of systems of blanks, should we not stress, for example, the importance of studying the structure of symbolic, detached languages; without undue regard, that is, to their content? Can we not find more *constructive* descriptions of the nature of mathematics?

Be that as it may, teachers who wish to spell out in detail the basic logical interrelationships of the real numbers will enjoy reading this book. —*Carl H. Denbow, Ohio University, Athens, Ohio.*

Understanding and Teaching Arithmetic, E. T. McSwain and Ralph J. Cooke, New York, Henry Holt and Company, 1958. Cloth, xi+420 pp., $5.50.

The purpose of this book, as stated by the authors in the preface, is to serve as a "teaching guide to teachers and prospective teachers who desire to improve their understanding of the meanings, vocabulary, and mathematical operations that constitute the language and science of arithmetic and who want . . . methods and materials . . . that may motivate and assist pupils in experiencing purpose, meaning, interests, and satisfactions from their study and use of arithmetic."

These are laudable aims. The pattern of presentation is laudable, too. So also is the aim of the authors to use each chapter "as an opportunity to think with teachers about the topics which are found in a typical curriculum in arithmetic," and to stress meanings.

Unfortunately, these aims are not achieved. The discussion is wordy, confusing, and full of inaccurate statements. The level of mathematical understanding and accuracy may be indicated by a few typical excerpts:

1. "When a person wants to record an amount he uses a number. When he desires to compare two numbers he uses subtraction." (page 6)

2. "A dividend such as zero cannot exist because a dividend notates a sum." (page 105)

3. "Multiplication is a thought process of finding the sum or product of a given number of like integers or fractions and notating the total with one number (the product)." (page 182)

4. "But in statements such as 'The population increased by 6%' there is little meaning because the whole quantity, i.e., former population, is not given. Unless this is known, the population increase cannot be computed." (page 240)

In quotation 1 the first sentence is obvious, the second false. One is at least as likely to use division to compare two numbers. Concerning 2, it is perfectly possible for zero to be a dividend; it cannot be a divisor. (The reviewer has no idea of what "notates a sum" is supposed to mean.) As to 3, multiplication has nothing to do with finding sums. And finally, the quoted statement 4 has a precise and significant meaning. There are many occasions when it is quite sufficient to know rate of increase without knowing the absolute amount.

The pedagogical aspects of the book seem more meritorious: the "explanatory questions," the suggestions for teaching procedures, the questions for self-evaluation, the "suggested activities," and the exercises all seem generally appropriate. But it is difficult to see how an adequate pedagogical superstructure can be erected on so shaky a mathematical foundation.

The strictly computational aspects of the book are substantially correct, though so many special cases must surely tend to confusion rather than to clarity of thought. The chapters

on measurement, problem solving, and evaluation are generally good.

The book includes also an extensive bibliography and a "basic vocabulary." The latter includes a number of terms of little importance like "abacus" and "decade combination," and others quite inaccurately defined: 0 is omitted from the list of Hindu-Arabic numerals, and is not a "digit"; a degree is called "a unit for measuring an angle or temperature" (this gives the impression that the same unit is used for angles as for temperature); multiplication is "a mathematical and mental process of rapid addition of a given number of like numbers."—*Albert E. Meder, Jr., Rutgers, The State University, New Brunswick, New Jersey.*

Letter to the editors

Dear Editor:

In his article, "Breakthroughs in Mathematical Thought," in the January 1959 issue of THE MATHEMATICS TEACHER, Professor Howard Fehr of Teachers College says "yes" to the following question: Has the new M.I.T. physics course been the work of college professors who have ignored the high school physics teacher? As one of the high school teachers working with the Physics Project, I know many other high school teachers who have been working with it. Professor Fehr has been misinformed. The new Physics Project has co-operated closely with secondary teachers and it is a daring affirmation of confidence in today's high school physics teacher.

This physics course originated as a scintillation in the eye of Professor Jerrold Zacharias of the Massachusetts Institute of Technology. The first co-ordinated attempt to produce a text and associated materials took place in the summer of 1957 when physicists, school teachers, and a variety of other specialists worked for the entire summer in Cambridge. *During that initial summer and throughout the work of the project since then more than 40 per cent of the academic staff has consisted of secondary-school teachers.* These teachers have worked on the writing, editing, designing of the laboratory program, and every other part of the Project's activities.

During the 1957–58 school year the first version of the course produced by the Physical Science Study Committee (PSSC) was tried out in eight schools. From that time on the experiment has received continual feedback from the high school teachers trying out the course.

During the 1958–59 school year about 270 secondary-school teachers are presenting the new course to their students. Many more teachers wanted to start the course but the Project was unable to accommodate them. The jump to 270 secondary-school teachers is a clear indication of its reception by teachers. (For all of its groups recommending and pioneering new programs in mathematics, the mathematics education world has yet to produce a course which has the impact on secondary-school mathematics teaching that the new physics course has had in two years on physics teachers.)

Another of Professor Fehr's charges against the new physics course is that it is applicable only to the brightest few per cent of the high school population. Professor Fehr's opinion here can be based only upon his own examination of the text. He has underestimated the average American high school student. Educational Testing Service is carrying out a complete testing and evaluation program for the new course. Data which is just beginning to come through from ETS indicates that the new course has surprising value for students of moderate ability. The results of complete data from ETS will be made available in future publications.

In the meantime, let me add *my* subjective opinion as a teacher. If I were given a physics class of gifted high school students, I would want to use the new course. If I were given an average class, I would insist upon the new course.

Sincerely,
DANIEL A. PAGE
Physics Teacher
University High School
Urbana, Illinois

Dear Editor:

More about Micky. I found it stimulating to read the two letters you printed, together with author Keedy's reply, in the February issue of THE MATHEMATICS TEACHER. The letters reveal a fact which is too often unknown—the fact that mathematicians and teachers of mathematics are human.

As an editor, I have always considered a flood of comments as a flood of compliments, for teachers must read what is published before they can pass along their ideas.

One man's opinion: THE MATHEMATICS TEACHER is the finest professional publication in print. More than any other single factor, it is contributing to the bettering of the teaching of mathematics in the United States. Keep up the excellent work.

We look forward to having you with us in Dallas next month.

Most sincerely yours,
KENNETH MANGHAM
Editor, The Texas Mathematics Teacher
Central High School
San Angelo, Texas

• TIPS FOR BEGINNERS

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan and William C. Lowry, University of Virginia, Charlottesville, Virginia

Oral testing

by Harry Wolff, Berlin High School, Berlin, Wisconsin

One of the important aspects of a complete evaluation program in mathematics is that of systematic oral testing. Every time we ask a question of a class or direct one to a particular individual we are, or should be, trying to discover some particular facet of the educational or learning process. Since oral questioning is a form of testing, I suggest that we give it the same careful consideration which characterizes our attempts to construct written tests. If oral testing is given conscientious planning and if it is applied systematically, it will easily establish itself as a functional and essential part of our evaluation program.

Of the many beneficial results to be obtained from oral testing, I think that two are especially suited for consideration in this article. First, oral testing presents the student with an opportunity to verbalize his mathematical ideas; second, it reveals weaknesses and strengths which would remain undetected by other types of testing. By assumption we are trying to develop individuals who can think originally and imaginatively in problem-solving situations. In my opinion this implies that one must be able to communicate his ideas to others. By permitting the pupil to verbalize his mathematical thoughts we are allowing him to perfect this vital segment of his education.

As a beginning teacher I remember being confronted many times with the reply, "I know what it is, but I can't explain it." This I felt could be attributed to the lack of experience the student had in verbalizing his ideas. I realized that too many times my questions were framed to initiate a positive or a negative reply only. This placed the student in the position of having a 50 per cent chance to render a correct decision even though he may not have understood the situation at all. As I analyzed my questioning technique, I detected another detrimental factor. By overuse, I systematically abused the "helpful" question. This is the type of question which contains so much information in the premise that the desired response is obvious to all. Although "helpful" questions and questions answered by "yes" or "no" are essential in any organized system of questioning, they should be minimized since they do not encourage the student to put into words any ideas or notions he may have concerning the subject being scrutinized. To accomplish this I felt it was necessary to revise my questioning technique.

Basically, the technique of quizzing for verbalization is that of asking questions which compel the student to organize his thoughts and present reasons for his contentions. "What evidence can you produce to support your conclusions?" "How can you defend your idea?" "What reasons do you have for believing that?" The questions should be part of a flexible pattern because the unique strength of oral quizzing lies in the fact that it can be used to seize instantaneously an idea and extend it with successive related inquiries. The questions should be designed to initi-

ate a thoughtful response, not a stereotyped reply. The following example is offered to illustrate several of these ideas.

An algebra class had an overnight assignment of a group of exercises of the form $ax=b$ where a and b were integers. The class had just become acquainted with the application to equations of the four basic axioms, and the assignment was designed by the author of the text as drill work in the use of the division axiom. The last exercise in the group ($2/3x=8$) was different from the others in that the coefficient of x was a common fraction. I anticipated some difficulty with this exercise so I had one student put his solution to the problem on the board. He solved it in the following manner and checked his result by substitution.

Method 1

$$2/3\,x=8$$

$$D_{2/3}\quad x=\frac{8}{2/3}$$

$$x=\overset{4}{\cancel{8}}\cdot\frac{3}{\underset{1}{\cancel{2}}}=12$$

A single question made it apparent that a vast majority of the class had solved it in this manner. A second student remarked that he used a different method:

Method 2

$$2/3\,x=8$$

$$M_3\quad 2x=24$$

$$D_2\quad x=12$$

And a third suggested that he solved it by "thinking" of the number which when multiplied by 2/3 gives 8:

Method 3

$$2/3\,x=8$$

$$2/3\times ?=8$$

$$2/3\times 12=8$$

$$x=12$$

Further questioning failed to produce any other approaches to the exercise.

I was intent on bringing out the idea of multiplying both members of the equation by the reciprocal of the coefficient of x. My first year of teaching I would have proceeded to suggest that method, demonstrate it, and explain it. Now, I posed this question: "Let us assume that we cannot use the division axiom to solve this equation. Can anyone find a method of solving it using only one or more of the other three axioms?" After a few moments of concentrated study, one student suggested that it could be solved by multiplying both members of the equation by 3/2. I asked the student to put his solution on the board with the others.

Method 4

$$2/3\,x=8$$

$$M_{3/2}\;3/2\times 2/3x=8\times 3/2$$

$$x=12$$

My first year, I probably would have led a discussion of the relative merits of the various methods at this point. Now I asked, "Will you please explain exactly how you arrived at this 'magic number' 3/2?" He replied: "Well, since the fraction is connected mathematically to the x by multiplication, the use of the addition and subtraction axioms is out. That leaves only the multiplication axiom because we agreed not to use the division axiom. Now, since I wanted to get one x (one times x) I knew I had to multiply some number times 2/3 and get the product one. I could see that the number had to be greater than one and less than two. I figured that it had to be a fraction with a numerator of three so the threes would cancel when I multiplied and it had to have a denominator of two so the twos would cancel. Then I multiplied the second member by 3/2 to keep the equation balanced." I complimented him and then said, "You mentioned something that interests me a great deal. How could you tell that the number you were

seeking had to be greater than one and less than two?" He replied that he could estimate the size because "one times 2/3 gives a number less than one and two times 2/3 gives a number greater than one," and he wanted the product to be "exactly one." To avoid ignoring the remainder of the class, I deferred a similar question concerning the way in which he ruled out the addition and subtraction axioms to a private discussion during the supervised study period. At this point I extended the idea by developing the concept of a reciprocal.

Another student volunteered still a fifth solution and this one also used only the multiplication axiom. He reasoned that if two thirds of x was equal to eight, then one third of x would equal four, and three thirds of x would equal twelve. On the board it looked like this:

Method 5

$$\begin{array}{lr} & 2/3\,x=8 \\ M_{1/2} & 1/3\;\;x=4 \\ M_3 & x=12 \end{array}$$

Referring to the five methods visible on the board, I asked the following question. "Which of these methods is the *correct* one for solving this type of equation?" With only slight hesitation, about a dozen individuals volunteered answers. As each method was presented, I countered with, "Why?" The following are some sentence summaries of the reasons given for selecting a particular method:

"Method 1 is the correct one because all the other exercises in the group were solved using the division axiom."

"Method 4 because it shows that the multiplication axiom can replace the division axiom."

"Method 2 because the axioms were all illustrated using integers and this method uses integers in two successive steps."

"Method 3 can only be used in special cases because if the fractional coefficient is a 'really weird' fraction you cannot guess the number easily."

"Method 5 is just like 4 except in 5 you do it in two steps instead of one."

Soon, the more critical thinkers began to realize that the word "correct" was the bottleneck and made the whole question senseless from their point of view. Confronted with this fact, I admitted, with mock humiliation, that the word "correct" did make the question rather meaningless, and I suggested that the word "convenient" would probably have been a better word. I restated the question using the word "convenient" and we continued.

Various reasons such as those presented by the students form the basis of some real teaching for me. They point up misunderstandings; they illuminate areas which require reteaching; they reflect the almost reverent faith of the student in the wisdom and discretion of the author of the text; they give me an insight into the depth of understanding of individual students. I follow up one of these discussions with personal individual contacts during the supervised study period at the end of the class or outside the class if at all possible. During these conferences I try to get the student to elaborate on his ideas and discuss his weaknesses and strengths.

Another result to be considered, the unique way in which oral testing reveals weaknesses, can best be illustrated by an example. In a general mathematics class we were working with per cent. One of the techniques discussed was to change common fractions to equivalent fractions with denominators of one hundred and then convert the hundredths to per cent. On a short quiz I gave the following exercises to be worked in that manner: (1) 3/10, (2) 2/4, (3) 4/16. I included the unreduced fraction, intentionally. One student produced the following solutions:

(1) $$\frac{3}{10}=\frac{33\frac{1}{3}}{100}=33\tfrac{1}{3}\%$$

(2) $$\frac{2}{4}=\frac{50}{100}=50\%$$

(3) $$\frac{4}{16}=\frac{25}{100}=25\%$$

During an oral quiz, I asked him to work the first exercise and indicate all the mathematical steps he used. He did it in this manner:

$$\frac{3}{10}=\frac{?}{100}=\frac{33\frac{1}{3}}{100}=33\tfrac{1}{3}\%, \qquad 3\overline{)100}^{\,33\frac{1}{3}}$$

I then asked him to work the other two and, as I sat aghast, he proceeded to solve them in a similar manner,

$$\frac{2}{4}=\frac{?}{100}, \quad 2\overline{)100}^{\,50}, \quad \frac{50}{100}=50\%$$

and

$$\frac{4}{16}=\frac{?}{100}, \quad 4\overline{)100}^{\,25}, \quad \frac{25}{100}=25\%,$$

producing the accepted answers, 50 per cent and 25 per cent. I learned two things immediately. First, the student knew very little about equivalent fractions even though he had two of the three exercises scored correct; second, there was a need for me to consider my quiz exercises more carefully in the future.

The following day I discovered a third fact. The student, after a discussion of the coincidence of the apparently correct solutions for the second and third exercises, became fascinated with the notion and after investigating it came to me with two more fractions, 3/9 and 5/25, which "worked with his method." Although this student knew very little about squares and square roots, he guessed that "if you divide the denominator by the numerator and get a quotient the same as the numerator, my method will work." A very simple algebraic analysis helped him to confirm this thought:

$$\frac{a}{a^2}=\frac{1}{a}=\frac{\frac{1}{a}}{1}=\frac{\frac{1}{a}\times 100}{1\times 100}=\frac{\frac{100}{a}}{100}$$

$$=\frac{100}{a}\ \%$$

This example, I believe, illustrates the way in which oral testing brings out facts which might otherwise remain obscure or unknown. This example also reveals the manner in which I usually employ oral testing. I find that I can apply this technique systematically by using it as a follow-up or as a supplement to my written tests and quizzes. Systematic use means regular use and, like most teachers, my problem is to find time to meet with the students individually. I accomplish a great deal by employing the individual oral quizzing during my supervised study period which normally occupies about one quarter to one third of each class period. Certainly oral testing is one of the most valuable diagnostic and teaching techniques I have at my disposal.

The British Minister of Education, Geoffrey Lloyd, paid tribute to America's method of educating its average children during his recent visit to the United States. He said, "We British have a very advanced system, we feel, for educating our clever children, but we also want to do more for our average boys and girls. We think you people have done a fine job in this field."

Dear Editor:

I read with interest the account of the seventh Polish mathematical Olympiad in the December issue of THE MATHEMATICS TEACHER. I wonder what comments this article caused.

What would our better secondary-school mathematics students do with the preparatory round of problems? Has the teaching of Euclidean geometry in this country been influenced in recent years by the multiple-choice type of question so prevalent on college entrance examinations? Such questions cannot test the student's ability to demonstrate an original theorem or construction problem. I think one of the most important benefits gained from the old-fashioned geometry course (with a good teacher) was the training received in recognizing in a new theorem elements with which one was already familiar. Hence the solution.

Problem 4 was new to me and, in hopes that you may have some other proofs of this theorem sent in (it would be interesting to see them), I am enclosing an outline of the first proof that suggested itself to me (after about ten minutes of analyzing the problem).

Theorem: The midpoint of the line joining the center of the circle inscribed in a triangle to the center of one of the escribed circles of the triangle lies on the circumscribed circle of the triangle.

Let $CA=b$, $CB=a$, $AB=c$.

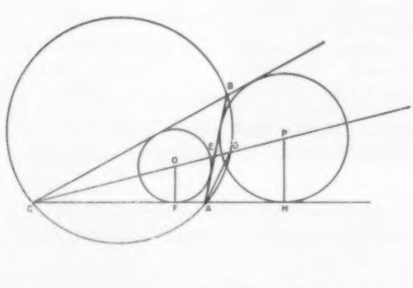
Let circles O and P be the inscribed and escribed circles respectively, both circles tangent to side AB of triangle ABC. Let the circumscribed circle intersect OP at D.

PO produced passes thru C, since O and P together determine the bisector of angle BCA.

Triangle CDA is similar to triangle CBE since angles CDA and CBE are measured by the same arc, and angle ACD is equal to angle BCE since CD is the angle bisector.

(1) From the similar triangles,

$$\frac{CD}{AC}=\frac{CB}{CE}.$$



By the well-known theorem for the angle bisector of a triangle,

$$\overline{CE}^2=ab-\frac{abc^2}{(a+b)^2}.$$

Therefore from (1) we have

$$(2) \qquad CD=(a+b)\sqrt{\frac{ab}{(a+b)^2-c^2}}$$

$$(3) \qquad CO=\sqrt{\overline{OF}^2+\overline{CF}^2}.$$

OF is the inradius and

$$=\frac{1}{2}\sqrt{\frac{(a+b-c)(a+c-b)(b+c-a)}{a+b+c}}$$

while

$$CF=\frac{a+b-c}{2}$$

by the theorem on equal tangents, so that CO becomes, upon substituting in (3) and simplifying,

$$CO=\sqrt{\frac{ab(a+b-c)}{a+b+c}}.$$

$$OD=CD-CO$$

$$(4) \qquad =(a+b)\sqrt{\frac{ab}{(a+b)^2-c^2}}-\sqrt{\frac{ab(a+b-c)}{a+b+c}}$$

$$=\frac{c\sqrt{ab}}{\sqrt{(a+b)^2-c^2}}.$$

By drawing a line through O parallel to FH, it is easily seen by similar right triangles:

$$(5) \qquad \frac{OP}{FH}=\frac{CO}{CF}.$$

By equal tangents it is seen that $FH=c$, so that using (5) and values already found,

$$OP=\frac{c}{\frac{a+b-c}{2}}\sqrt{\frac{ab(a+b-c)}{a+b+c}}$$

$$=\frac{2c\sqrt{ab}}{\sqrt{(a+b)^2-c^2}},$$

which is 2 OD as found in (4).

Very truly yours,

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NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Secondary Mathematics Curriculum

Report of the Secondary-School Curriculum Committee of the National Council of Teachers of Mathematics

A PARADOX of the present-day American classroom tends to characterize twentieth-century mathematics, a subject pulsating with the challenge of new ideas, as one that was permanently fashioned in the matrix of pre-seventeenth-century thought. It is quite true that mathematics, "the fastest growing and most rapidly changing of the sciences," does find peculiar strength in the time-tested validity of much of its traditional subject content. Rather than creating a paradox, however, this fact should, in actuality, provide background to give texture to the rich landscape of modern mathematical creation. The source of the paradox has not been in the subject-matter content but in the failure of teachers and curriculum planners to recognize the basic interrelation existing between the old and the new in mathematics. There has been failure to recognize that not only can the excellence of the old give depth to the perspective of the new, but also that the elegance of the new can add refinement to the interpretation of the old.

Recent events are tending to eradicate even this paradox. Increasing emphasis is being given to the significance of mathematics as a substantial part of our culture and to the rapidity with which it is changing and is producing change. As a consequence, both lay opinion and professional opinion have become quite concerned over the curriculum content and the instructional practices which characterize the current mathematics program in the schools of our nation. The significance of this concern and the serious nature of its implications are underscored by the questions which teachers and administrators are asking.

What is the place of mathematics in a changing society?

What are some of the more significant new interpretations and uses of mathematics?

What are some of the more recently developed areas of mathematical subject matter from which pertinent adaptations to the secondary level of instruction can be made?

What are the criteria to be used in planning a program in mathematics that will guarantee opportunity for maximum benefit both to the slow learner and to the mathematically gifted?

What are the criteria to be used in planning a program that will provide equal assurance for appropriate functional competence both to the terminal student and to the college-capable student?

How can we secure better counseling of high-school pupils so that they may be informed about avenues of potential interest and challenge in mathematics as well as warned against possible frustrations and disappointments in the unwise attempt to reach unattainable goals of mathematical achievement?

Which of the conventional topics of mathematics, if any, should be radically changed or eliminated?

Which of the newer developments in subject matter have become of more than mere specialized significance?

What of the new can be used to enrich the traditional, and what are approved procedures for the accomplishment of such enrichment appropriate to secondary-school pupils?

What of the old can be used to orient and clarify the new in such a manner that secondary-school pupils may profit most from experience with the new?

Questions such as these emphasize the urgent need for a searching reappraisal of the place of mathematics in our changing society, the content of the mathematics curriculum in our schools, and the instructional procedures used to help the pupils in our schools derive the greatest benefit from this curriculum.

Orientation of the curriculum

Our changing society. The perspective to be gained from consideration of the gross attributes of society at present, and the changes occurring in them, can help us comprehend the need for certain emphases in education in general and in the study of mathematics in particular.

The recorded history of man on earth almost vanishes if one looks back 100 life-spans. For example, one could receive a message from Plato that need not have passed through the mouths of more than thirty-three men. For Jesus, twenty-eight men would do, and for Mahomet, nineteen. Even these startling calculations do not reflect adequately the explosive character of the development of civilization and knowledge. Word could have come from Gutenberg through a chain of seven men, and from Newton through four. Two life-spans ago the steam engine and the lathe, keys to the industrial revolution, came into general use, and the internal-combustion engine came only one life-span ago. The rate of development during our lives has become fantastic, though we tend to become blind to it; e.g., it took us only a couple of months to become blasé about man-made satellites. *Thus most of what man knows and has accomplished is incredibly recent.* We are living quite literally in the presence of an explosion of knowledge.

We are also living in the presence of a biological explosion, some evidence of which we see in our classrooms. The population of the United States, for instance, is increasing now at the threefold per life-span rate; this results from a medium birth rate and a low death rate, and is exclusive of immigration. *Most of the groups in the world have birth rates which will put them in the fourfold to sevenfold class if their death rates are reduced substantially—which can be done quickly.* The limit of the explosion rate is supposed to be about sevenfold per life-span, when not seriously impeded by negative forces, such as shortages, pestilence, and social taboos. If the above estimates had dealt only with people who had lived to be adults, the explosion rates would be greatly increased, and it would be evident that the adults now alive constitute a very interesting fraction of all who have ever lived. Even so, we should look on ourselves as the precursors of mankind, for the great mass of men is just now arriving. For example, the *net increase* in humanity during the gestation period that ended today is numerous enough to populate the sixteen largest cities of the United States.

The present attainments of the men of our planet are remarkably varied. Some have domesticated only the dog, while others at least can equal our most recent achievements. Furthermore, it is perfectly feasible for any given group to advance as much during one life-span as the leaders of the race did in ten. As a matter of fact, some of our children absorb the essence of 100 life-spans and then break new ground, in a fraction of a life-span. *There are several large groups on the planet now which have the vitality to surge beyond us during the next life-span, and which almost certainly will do so if most of our people seek solely to enjoy the status quo.*

The men of this planet are also alarmingly fierce. Indeed, we find it expedient within the confines of our own quiet soci-

ety to make perhaps one in every one hundred able-bodied males a policeman. Possibly even more alarming is the fact that the advanced and powerful nations of the earth have within our life-span demonstrated that their capacities for cruelty and horror are the equal of anything in history. *So if some new group surges into the lead, it will not be astonishing if it happens to obliterate us in the process.*

Until now men have generally lived and died in a small number of fixed groups in fixed places. Their principal contact with the rest of the world has been in the form of friction with the few groups in their immediate neighborhood. Within our life-span, however, the internal-combustion engine has been combined with the air as a medium of transportation, and *the global properties of our planet have become of profound importance.*

Since we found the life-span useful in discussing long periods of time, let us try the *breath-of-life* as a unit for short ones. *It is a literal fact that, from now until eternity, no man on earth will ever be able to guarantee that he can breathe one thousand times before he dies by violence.* A thousand breaths is not very many, and this large number applies *only* if one's destruction is set in train by the most distant man on earth, for it corresponds to the flight-time of a ballistic missile for a range of 12,500 miles, one-half the distance around the earth. It would be very difficult to build a missile which would take a longer flight-time, so we should not expect a lucky extension on the thousand breaths. The people of the earth in general, and of the United States in particular, do not seem to be aware of this fact. It is difficult to comprehend what will arouse them to an awareness of the situation as it really exists.

An inward look at our society reveals that it is rich in choices for the individual, in its physical environment, and in its organization of men and machines for the creation of various forms of wealth. There are today many accomplishments of man that may rank, or even outrank, the steam engine and the lathe as key elements in the story of human technological development. Automation is but one concrete example of such a key element in the explosion of knowledge that is taking place.

The electronic computer, for example, under intensive development only since World War II, already reaches speeds of tens of thousands of operations per second with high reliability, and may carry within its memory the instructions needed for an almost endlessly long and complex sequence of operations. Furthermore, such a machine can be used to govern the behavior of another machine. Thus, rather than use a costly special-purpose machine to do some part of a manufacturing process, one may use a more versatile machine controlled by a computer to do more of it. And, by changing the instructions in the computer, one may use the machine tomorrow to make a different product. The machine can perform elaborate operations as well as simple ones, and it is feasible to use it for quite short production runs—in fact, even to construct prototypes. *Thus the impetus toward mechanization is suddenly ferocious.*

In certain activities that require brute power, machines have tended to reduce the need for unskilled workers. Similarly, in activities requiring simple repetitive work, machines have tended to reduce the need for semiskilled workers. Simultaneously, the needs for professional and managerial skills—to improve the product, the plant, and the process—have increased, but not by a like amount. The machines of the immediate future will continue these trends but at an accelerated pace. Up until now the growth of new industries has kept in fair pace with this mechanization, as well as with the natural increase of the labor force. It is not clear that this factor will always be able to achieve the happy result of balancing these two explosive forces; in fact, it is downright unlikely.

An impressive picture of the potential

of the effect of automation on our society may be obtained from the following descriptive occupational profile: at the left end of the horizontal scale place those occupations that depend upon the muscles as sources of power, and at the right end, professional occupations that demand training and versatility. Then choose as ordinates the percentages of the employed population engaged in the respective occupations. Such profiles drawn at intervals of an *occupational life-span*, say every forty to fifty years, would emphasize an upward trend from left to right that is taking place at a phenomenal rate. An attempt at extrapolation for only one occupational life-span (when the present school population will be nearing the end of their occupational careers) brings to light a picture of alarming implications.

New interpretations and uses of mathematics. One of the distinct and important factors contributing to the great explosion of knowledge which has been taking place during our life-span is the over-all revolutionary advance in the uses of mathematics. Theoretical mathematicians have never produced more new ideas, new theories, and new potential for great breakthroughs in all branches of science. The astonishing developments in the physical sciences are continually creating demands for new interpretations and uses of mathematics. Of possibly even greater significance in this revolution are the demands which are coming from new users of mathematics.

The life sciences, the schools of business administration, and the social sciences are increasing their demands for the use of mathematical principles and techniques. The biologists are applying information theory to studies of inheritance; men from business and industry use the techniques of operations research in scheduling production and distribution; psychologists and sociologists make many uses of both the elementary and the more sophisticated properties of modern statistics; the analysts of human behavior are finding major assistance in the principles and properties of game theory. The demands made by such important new uses and interpretations of mathematics cannot be satisfied by the provisions of the classical programs that originally were dictated largely by the engineering schools.[1] In recent years the engineers also have broadened their concept of what constitutes an adequate background in mathematics for a successful engineering career.

This revolution in the interpretation of the role of mathematics as an important element of our social structure is closely related to the evolution of the electronic computing machine, but it is even more firmly woven into the fabric of our social order. The new uses of mathematics place great emphasis on its basic structure and on its function as a language in terms of which theories and hypotheses can be precisely formulated and tested. Rather than the manipulation of formulas and equations, the measurement of configurations, and the performance of computations, the principal contribution of mathematics is fast becoming the construction of mathematical models of events, whether actual or hypothetical. Through the use of such models the obscuring details and unessential trappings of problem situations can be removed so that the essential pattern can be displayed clearly for thoughtful analysis. The demands of our technological society tend not only to emphasize the need for a stronger foundation in the fundamentals of mathematics but also to require the reconsideration of the subject-matter content of this essential foundation.

[1] The "classical secondary program" has generally been considered to consist of two years of algebra, one year of plane geometry (strictly Euclidean), one semester of solid geometry, and one semester of trigonometry (primarily numerical). The "classical college program," while not quite so easily defined, frequently may consist of three semester hours of college algebra (not at all of standardized uniform content), four to six semester hours of analytic geometry (plane and solid), and six to eight hours of calculus (differential and integral). In many regions of the United States trigonometry is transposed from the secondary to the college program; in other regions, the college program begins with analytic geometry or calculus.

The many new uses of mathematics not only have increased the demand for mathematicians, but have placed new emphases on the type of training they need. Even the requirements for programs of "minimum essentials" in mathematics at different levels of attainment need to be subjected to critical reconsideration. This is true whether the evaluation is made in the context of the needs of the lay user of mathematics or of the individual looking more specifically toward professional uses of mathematics. Thus we find that, following a half-century of relative stability, the content of the mathematics program at all levels of instruction is now undergoing a wide variety of changes. We speak of traditional and modern mathematics, and have erudite discussions on the extent and pace at which each should be developed in our curricula. We are experiencing a rash of new courses involving a wide variety of combinations of old and new material. In the secondary field we have, at one extreme, courses whose main emphasis is on topics such as group theory and the number system and, at the other extreme, courses in which the emphasis is in rushing through the classical topics so as to find time to include a year or more of traditional college mathematics. At the college level the variety of courses is even greater.

Three important factors can be identified in this mathematical revolution that is taking place: (1) the broad variety of mathematical models and situations in which they are employed; (2) an increasing recognition of and attempt to deal with the stochastic nature (*i.e.*, the group behavior characteristics) of the world in which we live; and (3) the rapid development of the computing machine.

Each of these factors has important implications for the mathematics teacher. The increased employment of mathematical models brings into perspective the importance of the foundations of mathematics. There is a new urgency for helping the student obtain a sound understanding of such mathematical concepts as function and relation. Elementary set theory provides the language for describing models. Indeed, a good case can be built for the thesis that set theory is the most universally applicable part of mathematics. The recognition of the stochastic nature of our world results in an increasing pressure for early introduction of probability and statistics into the mathematics curriculum. The power of the computing machine in solving linear equations has encouraged the development of many linear models and thus places a new requirement for linear algebra in our elementary courses. Algebra now takes on importance in its own right, not just as something needed as an aid for manipulations in calculus. The computing machines have vastly increased the demand for mathematicians at all levels: coders, programmers, and numerical analysts are in heavy and steadily increasing demand.

The unprecedented demand for mathematicians, their need for a new and more intensive training program, and the basic mathematical requirements of our technological age combine to pose a very difficult curriculum problem that demands the thoughtful attention of those whose concern it is to provide the most effective program in mathematics for our secondary schools.

Mathematics programs in European countries. There are several reasons why it is difficult to compare the mathematics programs of European countries with that in the United States. The fundamental principles giving structure to the program of educational endeavor are quite different. One basic factor in the European philosophy is that the educational program should provide a sifting process that will allow only the very able in academic learning to continue upward into advanced study. Another factor leads to basic differences in that part of the educational structure concerned with starting age, manner of teaching, length of study, and prescribed courses. (In European schools

there are no electives.) A third factor is earlier and greater specialization. A fourth factor is concerned with the training and selection of teachers, which is generally more severe in European countries than in the United States.

Generally speaking, in Europe a pupil begins his secondary education at the end of the fourth or fifth grade, age ten or eleven years. At this time he is one of the upper 15 per cent to 30 per cent in academic ability—the other 85 per cent to 70 per cent continue an elementary program for several more years. The subject matter he studies from age ten to seventeen or eighteen is, in general, the same as that taught in our four-year high schools and the freshman college year. It is built, however, in a structure of a greater degree of difficulty, complexity, and application than in the corresponding programs in most sections of the United States. Also, algebra, geometry, and arithmetic are pursued continuously through three or four years and not organized in distinctly compartmentalized full-year programs. The European approach to trigonometry, analytic geometry, and calculus is likewise an integrated one.

This highly selective intellectual program has resulted in two serious problems: (1) the lack of sufficient numbers of trained scientific and professional persons to carry on the demands of modern technology; and (2) a rigorous classical treatment of mathematics that makes it very difficult to include any innovations, reforms, or changes in the present program even though such changes are proving to be very desirable. The centralization of the school system and the standardization of courses by means of examinations also contribute heavily to the bringing about of this latter situation.

Thus, the European countries are facing problems in mathematics education quite similar to those we face in the United States, and, as yet, they have not advanced as far as we have in getting at possible solutions that have the potential of producing a better mathematics program. The Organization for European Economic Cooperation (O.E.E.C.) has recognized the seriousness of the problem and is seeking a way to introduce more modern notions into the secondary mathematics and science programs as well as to increase the number of pupils pursuing the program. To this end the O.E.E.C. has initiated a "Survey of Current Practices and Trends" in all of its participating countries, which include the United States.

This survey will be followed by an International Seminar on modern mathematics and the implementation of the newer mathematics program into the program of the secondary schools. In this seminar, expression will be given to the combined thinking of leading mathematicians, educators, and teachers on what modern mathematics has to offer the secondary school and how teachers and curriculum planners can be helped to carry out the desired reforms.

In general, Europe has a good program for the intellectually elite, but it has yet to solve the problem of creating a challenging program for a large number of able pupils who fail to get into the restricted elite group. In the United States we have a good program for the large mass of able pupils and are faced with the problem of developing a program for the elite. Neither group should lose the desirable characteristics of their respective education programs. Each group must make the changes in the content and structure of its program necessary to put it in harmony with contemporary mathematical thought and to shape it for significant challenge to all educable individuals and, in particular, to pupils with special mathematical aptitude.

The formulation of objectives. What implications for change in our educational program are to be found in the explosive changes taking place in our society and in the revolutionary thinking about mathematics? How can we best prepare our children to meet the great problems and fundamental uncertainties of the future? Is it

not a good idea to provide them with opportunities to learn as much of the accumulated lore and wisdom of the race as they can absorb and to spare no effort to encourage them toward creativity? What criteria can be used as guides in selecting from the store of accumulated knowledge that which might be of most value to each child in future situations and roles, even those not presently within the child's interest or intent? What is the role of mathematics in such a program? In shaping a pertinent mathematics program, how can we best resolve the conflicting claims of classical mathematics and the newer topics for places in the curriculum? Answers to such questions as these are of the greatest significance. They should be sought only under the control and direction of a carefully constructed body of objectives of educational endeavor.

Many efforts to formulate such a list of objectives produce unsatisfactory results because the nature of the task is not clearly understood. Some persons view the task primarily as one of securing a list of subject-matter concepts and abilities. This procedure usually produces a rather lengthy list of specifics, such as triangle, parallelogram, bisection of an angle, the quadratic formula, difference of two squares. Other persons view the task as being primarily one of indicating certain types of desirable behavior, such as recall, understand, apply, appreciate, analyze, and reason logically. Such lists are usually rather short and fairly general.

While each of these points of view can lead to the identification of essential elements of a valid set of objectives, it can do no more than provide an incomplete guide to effective instruction. The task of setting up a truly significant list of objectives involves not only the specification of behavioral elements but also the specification of subject concepts and abilities and the establishment of relations between the two sets of elements.

There are three basic criteria that can be used as helpful guides in the formulation of an effective set of objectives. They are:

1. *The statement of an objective should indicate both the desired behavior and the type of situation in which it is to occur.*
2. *Objectives should be stated in terms of desired pupil behavior rather than teacher behavior.*
3. *Objectives should be formulated at a level of specificity such that for each objective it is possible to infer some learning activity appropriate for helping pupils achieve it, and also such that it is possible to devise means of evaluating the achievement, but not to a greater degree of specificity.*

The first step in the process of selecting objectives is to secure a general description of the *situation* in which the objectives are to be operative. Ordinarily, the structural level—elementary, secondary, or higher education—will be specified, and, frequently, other initial restrictions will be imposed to identify the situation. The second step is that of securing a *universe* of tentative objectives from which valid selections can be made. For mathematics, this universe must include those objectives believed important by competent mathematicians.

If the desired *behaviors* are used as the primary classification, objectives may be displayed in the familiar tabular form:

Behavior	*Sample content*
(1) recall	(a) addition table for natural numbers to $9+9$
	(b) the rational approximation of π to five significant digits (3.1416)
	(c) conversion factors for measurement systems; e.g., 3 ft. in 1 yard, 5280 ft. in 1 mile, 2.54 cm. in 1 inch
	(d) formulas; e.g., in the domain of complex numbers if $ax^2+bx+c=0$, and $a\neq 0$, then $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
	(e) definitions; e.g., congruence, logarithm, reciprocal
(2) understand	(a) meaning of addition
	(b) concepts of rational and irrational number, rational approximation

(c) role of standard units in measurement: utility of having units of different magnitude in a system
(d) algebraic principles used in expressing $ax^2+bx+c=0$ in the form

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

(e) meaning of congruence, etc.

The content lists associated with such behaviors as "recall," "understand," and "apply in new situations" are inevitably very long. It is in association with them that the bulk of the subject matter is specified in detail. In the case of most other objectives the content may be expressed in much more general terms. Consider, for example, the objectives in which the behavior is "to be interested in" or "to appreciate." For such objectives it is often sufficient to indicate broad categories of content.

Similarly, the fundamental objectives relating to problem solving, to be useful, must have the behavior specified in some detail. One such specification is the following: The student should be able to:

1. *Recognize* and *formulate* mathematical problems.
2. *Collect* and *organize* data relevant to the solution of problems.
3. *Analyze* and *interpret* data relevant to the solution of problems.
4. *Obtain* and *verify* solutions to problems.

Each of these behaviors can, of course, be defined more specifically. When this is done, such terms as "analyze" and "interpret" are seen to cover, or call for, mathematical concepts and methods, and many of the manipulative techniques are subsumed. In similar fashion, the more specific behaviors that are involved in both inductive and deductive reasoning are included. A complete specification calls for the listing of numerous relevant logical concepts and principles, and thus these become an essential part of the content.

At the same time, we note that it is unnecessary, and even undesirable, to attempt any specific listing of the problems to which these behaviors are to be applied. Any mathematical problem the students can recognize as a problem and formulate (or understand as a given formulation) is admissible.

This discussion has included only incidental mention of a kind of objective which in the past has dominated most lists. This objective may be defined as "the ability to operate with mathematical objects." When analyzed more specifically this includes the ability to add, subtract, and perform various other operations with numbers, vectors, sets, and other objects of mathematical thought. Normally, the behavior expected often includes skilled performance (speed and accuracy) as well as the mere ability to get the task accomplished. If the *ability* is to be accompanied by *understanding*, the relevant basic concepts and principles must be included in the subject-matter and *applied* to the operation. The commutative, associative, and distributive "laws" are important examples of such principles.

The general principles suggested above are designed to help local committees select sets of objectives that are valid for their particular situations. Although it is impossible to give here a comprehensive list of objectives that could be adopted as appropriate to varying local situations, we can indicate very briefly some notable readjustments in emphasis that are now occurring.

First, objectives that call for understanding of concepts and principles are being emphasized now more than ever before. As a result, there is *relatively* less emphasis on the objectives that call for the ability to recall items from extensive lists of information, and for skill in a wide variety of manipulative techniques. It is being widely recognized that a better balance of emphasis on these different types of behavior is long overdue.

Moreover, the sorts of concepts now being stressed are different. In the past many teachers have worked hard to get students

to "understand the vocabulary of secondary mathematics." Highly specific terms such as *coefficient*, *quadrilateral*, and *converse* have been defined and students have been "drilled" in stating the definitions. Today, there is increasing concern with a broader type of concept indicated by such terms as *structure*, *relation*, and *mathematical system*. The scope of the objective "understand important concepts" has thus been extended. When this objective is achieved, its impact on the total program is great, for these broader concepts serve to bring together a host of details and enable the student to see mathematics in an entirely different light.

Second, the content or subject-matter aspect of the objectives is being broadened and generalized by giving *explicit* attention to concepts which have up to now been implicit, "incidental," or largely ignored. Prominent among these are the concepts of *set* (and numerous associated ideas, such as "solution set of a sentence"), certain concepts and principles of logic, and the analytical properties of the trigonometric functions. Topics from analytical geometry and statistics, and other topics mentioned in later sections of this report, are recommended authoritatively for inclusion in the universe of objectives.

Third, there is much more emphasis upon giving students certain types of learning experiences that serve not only to enhance their understanding of mathematics, but also to enliven their interests and increase their appreciation. Experiences that lead to "discovery" of mathematical properties and relations are highly favored for this purpose. It should be noted that what is sought are certain desirable behavioral characteristics that are suggested but not adequately specified by such broad behavioral terms as *discover*, *be interested in*, *appreciate*. Such phrases as "wants more mathematics" and "wants to know more about mathematics" bring the nature of the objective into the open.

For many teachers of mathematics the most urgent immediate need is to become familiar with the concepts and content of the types suggested above, and with ways of teaching that elicit the desirable behaviors indicated. Then, and not until then, will they be in a position to formulate objectives that are valid when judged in terms of any reasonably comprehensive set of criteria.

The content of the curriculum

Implications of contemporary mathematics. One of the major problems in the improvement of the mathematics program in the secondary schools centers around the nature of mathematical thought in these grades. What are the implications of contemporary mathematics for the structure of the program for grades seven through twelve? What part should "modern mathematics" play in determining the objectives of the program, and in what way?

"Modern mathematics" for secondary schools is composed partially of new points of view toward traditional topics and partially of the replacement of a few traditional topics by new ones. The new points of view and the new topics make it possible to show the interrelations and extensions of the old topics. When there is a basis of concrete experience, abstraction not only adds insight to the experience but also shows gradually where the topics of mathematics fit into a unified whole. For example, instead of studying first plane geometry and then solid geometry, we study "geometry" with special emphasis upon the plane.

Mathematical systems are man-made. They evolve as models for the representation and interpretation of the physical universe. Thus, the physical universe provides a basis for pupil discovery and understanding of mathematical systems. At all levels of instruction more emphasis should be placed upon pupil discovery and reasoning, reinforced by greater precision of expression. For example, distinctions should be made between the operation "minus" and the quality of a number "negative"; between a "region" on a

plane and its "area"; between "radical" and "square root"; between any object and a name for that object; between "number" and "numeral" (a symbol for a number); between "precision" as indicated by the size of the smallest unit used in a measurement and "accuracy" as crudely indicated by the number of significant digits required in expressing a measurement. In all cases the development should occur as early as is compatible with the knowledge and preparation of both the teacher and the pupil.

New mathematical concepts should not be isolated as separate units. Rather they should be woven into the whole fabric of the curriculum. A few concepts will be mentioned briefly to illustrate the unification of old and new topics.

SETS: The concept of set can be used advantageously at all levels. Elementary-school teachers now use sets to develop number concepts; most secondary-school teachers should make still more effective use of sets as they extend these number concepts. At the elementary-school level the pupil views a set as a collection of elements, such as the birds pictured on a certain page of his text, the pencils in his hand, the members of his immediate family. In each case he can identify whether or not any specified element is a member of the set under consideration. This concept of a set and the members of a set suffices for many of the uses of sets in secondary-school mathematics. It also provides a basis for the two special sets and the two operations on sets that contribute most to the study of secondary-school mathematics. The two special sets are the empty set (sometimes called the null set), which has no elements, and the universal set, which contains all elements under consideration. The two operations with sets are union (giving rise to the set of elements that belong to at least one of two or more given sets) and intersection (giving rise to the set of elements that belong to all of the given sets). These concepts can be, and in many junior and senior high schools are being, used very advantageously. The formal theory of sets is not appropriate for secondary schools except perhaps in the twelfth grade. The language and basic ideas of sets can be used advantageously at all levels of instruction.

EQUATIONS AND INEQUALITIES: Statements of equality and inequality should be treated in juxtaposition. The use of sets as a unifying concept is illustrated by the following classification of equations in one variable. Briefly, an equation is a statement that two expressions, upon replacement of the variable by a numeral, represent the same number. The truth of an equation in one variable depends upon the value of the variable. Thus $x+3=5$ is true when $x=2$; false when $x=3$. The set of all possible values of the variable (usually the positive numbers, the rational numbers, the real numbers, or the complex numbers) is the *domain of the variable.*

The concept of *solution set* (the set of values of the variable for which a statement is true) is very helpful in any discussion of equations. An equation in x is an *identical equation* (*identity*) if it is true for all permissible values of x. It is a *conditional equation* if there are values of x, in its prescribed domain, for which the equation is not true. The solution set of a conditional equation may be of three types: (1) it may contain an infinite number of verifiable elements, (2) it may contain a finite number, or (3) it may be the null set. For example, if the domain of x is the set of real numbers, the equation $1+x=x+1$ is an identity; the solution set for $\sin x=1$ contains an infinite number of elements as does that for $x^0=1$; the solution set of $x+3=5$ is $\{2\}$, and of $x^2-x-6=0$, it is $\{-2, 3\}$; and the null set is the solution set of $x^2+1=0$.

The concept of a solution set is particularly useful when considering inequalities and simultaneous statements of equality and/or inequality. As an illustration consider the variables x and y, the domain of each being the set of all real numbers. The solution set of the equation $2x+y=8$

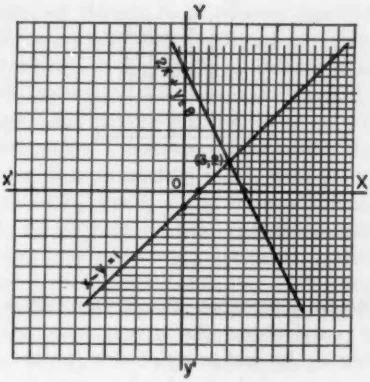

Figure 1

would be the set of all number pairs which are co-ordinates of points of the line labeled "$2x+y=8$" in Figure 1. An easily recognized subset of this solution set, $\{(0, 8), (3, 2), (4, 0)\}$, is indicated in the graph. Similarly, the solution set, and an easily recognized subset, $\{(0, -1), (1, 0), (3, 2)\}$, of the equation $x-y=1$ is indicated. When these two equations are considered simultaneously the solution set is the intersection of the two previously mentioned solution sets. It consists of a set with one element, $\{(3, 2)\}$.

Now consider the inequality $2x+y>8$. The graph of its solution set is the portion of the plane indicated by the vertical-line shading. Similarly, the graph of the solution set of the inequality $x-y>1$ is the portion of the plane indicated by the horizontal-line shading. When these two inequalities are considered simultaneously, the graph of the solution set is the portion of the plane indicated by the crosshatching.

The solution set of the simultaneous equations is a finite set consisting of the one element (3, 2). All other solution sets mentioned in this illustration are infinite sets and the graph is, in each case, an incomplete graph used merely as a helpful representation of the true graph.

PROPERTIES: The contemporary emphasis upon mathematical models requires that operations and their properties be stressed at all levels. Each of the operations of arithmetic—addition, subtraction, multiplication, and division—should be considered as associating a number with two numbers given in a specified order. The existence of the sums, differences, products, and quotients indicates the closure of the set of numbers involved under the operation used. For example, some quotients of integers are integers; some are not. The set of integers is not closed under division. The set of positive rational numbers is closed under division since the quotient of any two positive rational numbers is a positive rational number. The closure of each set of numbers (positive integers, positive rational numbers, integers, rational numbers, real numbers, complex numbers) should be considered informally at first, formally in the upper grades.

Each of the operations of arithmetic is concerned with two numbers given in a certain order. The commutative properties for addition and multiplication state that the order of the two numbers is unimportant in addition and multiplication.

$$a+b=b+a; \qquad ab=ba.$$

The order of the given numbers is important in subtraction and division, i.e., $a-b$ and $b-a$ are not necessarily equal; $a \div b$ and $b \div a$ are not necessarily equal.

Many elementary school pupils learn families of number facts, such as

$$3+2=5, \qquad 5-2=3,$$
$$2+3=5, \qquad 5-3=2$$

and

$$3 \times 2=6, \qquad 6 \div 2=3,$$
$$2 \times 3=6, \qquad 6 \div 3=2.$$

Notice this practical use of the commutative properties for addition and multiplication. Notice also the relationship between addition and subtraction,

$$(a+b)-b=a,$$

and the relationship between multiplication and division,

$$(a \times b) \div b = a, \qquad (b \neq 0).$$

Addition and subtraction are called inverse operations; multiplication and division are also inverse operations. The emphasis upon this relationship should be extended in the upper grades to include finding powers and roots of numbers and also finding logarithms and numbers having given logarithms.

The closure, commutative, associative, and distributive properties of numbers are dealt with informally in the elementary grades. These properties should be given more explicit and formal treatment beginning with grade seven, and algebra should not be taught without giving them explicit attention. The ideas should come before the terms, but if the reason for the separate terms is made clear, early introduction should be possible. In fact, throughout mathematics in the schools, where one can be precise in the vernacular without the use of technical terms, such terms should not be introduced; where technical terms are necessary they should be brought in without fear.

LOGIC: The use of reasons for statements should be common at all levels. Definitions and assumptions are as important in algebra as in geometry. Equation axioms and inequality axioms should be stressed.

Clear thinking is essential at all levels; formal proofs are for mature pupils. While a course in formal logic is out of place in the secondary-school curriculum, logic should be taught where appropriate throughout the period of training in mathematics. The students should become acquainted early with the meaning of "implies" and of the "converse," "inverse" (sometimes called "negation"), and "contrapositive" of a simple theorem.

USES OF SETS: Here is a brief outline which shows in sequential fashion how some of the terminology of sets might be woven into certain traditional material, and how this, in turn, could lead into

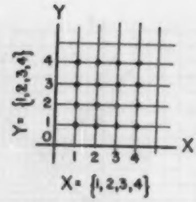

Figure 2

selected topics not usually appearing in the curriculum. As given, this would probably be inappropriate for junior high school, but if the teacher has some familiarity with such a development he could use some of these ideas, perhaps without all of the terminology, in his introduction of material.

First, the ideas of a set, union and intersection of two sets, the null set, and subsets could be illustrated by examples both mathematical and nonmathematical in nature. These illustrations would include from mathematics:

S_I = the set of all integers,
S_r = the set of rational numbers, and
S_R = the set of real numbers.

The set S_I is a subset of S_r since each element of S_I is an element of S_r. The set S_r is a subset of S_R; S_I is also a subset of S_R.

Then one might consider the set P_I of all ordered number pairs (x, y) where x and y represent integers (positive, negative, and zero), i.e., x and y are in S_I. The fact that the pairs are ordered means, for instance, that the pair (2, 3) is considered to be different from the pair (3, 2). There is a one-to-one correspondence between the set P_I and the intersections of the lines on a sheet of graph paper when each square has side one unit in length. The intersections are called "lattice points" and constitute the graph of P_I. For example, if x and y represent only positive integers less than 5, then the graph would be the intersection of the two sets $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2, 3, 4\}$, represented by the sixteen lattice points of Figure 2. One might also

consider the set P_r of ordered pairs (x, y) where x and y may represent any rational numbers; or the set P_R of ordered pairs of real numbers. The solution sets of the equations $2x+y=8$ and $x-y=1$ and their companion inequalities, discussed above, were all subsets of P_R. Also, since the symbol $a+bi$, where a and b are elements of the set S_R, represents all complex numbers, there is a one-to-one correspondence between the elements of the set of all complex numbers and the elements of the set P_R.

Important subsets of each of the sets P_I, P_r, and P_R may be obtained by restricting x and y by means of relationships. If the restriction is such that no two distinct pairs (x, y) have the same x, we have what is called a "function" of x. For example, if the restriction is $x^2=y$, the set of ordered pairs (x, y) satisfying this restriction define a function (even though two pairs having different x's may have the same y); if the restriction is $x^2+y^2=5$, the relationship is not a function of x since the pairs $(1, 2)$ and $(1, -2)$ belong to the set and have the same first element.

Such a set of ordered pairs might have a finite number of pairs, e.g., it might consist of four pairs: (1, 4), (2, 3), (3, 2), (4, 1)—all the pairs of positive integers whose sum is 5; e.g., all the ways of having five coins consisting of cents and quarters containing at least one of each kind. The graph would consist of four points.

Such a set might have an infinite number of pairs. One example would be

F_1: (x, y) where x and y are integers whose algebraic sum is 5.

The graph would consist of all lattice points on the line through (0, 5) and (5, 0). F_1 is a subset of P_I. One might also consider

F_2: (x, y) where $x+y=5$ and x and y are rational numbers.

(The brighter student might want to consider how close together these points might be on the line.) F_2 is a subset of P_r.

Another set would be

F_3: (x, y) where x and y are integers satisfying the equation $4x^2=y$.

Here F_3 would have no pairs whose second element is 5 or 9 or any negative number. What is its graph? F_1, F_2, and F_3 are all examples of functions. These sets still retain the property of being functions even if the only restriction on x and y is that they be real. Other sets would be concerned with inequalities, like $x+y\leqq 5$ or $4x^2\leqq y$. These are relations, not functions.

Other sets of number pairs are associated with simultaneous equations. For instance, consider F_1 above with

F_5: (x, y) with x and y integers for which $x+3y=13$.

The pair (1, 4) is the intersection of the sets F_1 and F_5 and its graph is the intersection of the graphs of F_1 and F_5. However, if we have

F_6: (x, y) with x and y integers for which $x+3y=14$,

there is no intersection of F_1 and F_6, i.e., the intersection is the null set. But if F_7 is the set determined by the equation for F_6 in which x and y are only restricted to be rational, then F_7 and F_2 have the intersection $(\frac{1}{2}, \frac{9}{2})$. On the other hand, if F_8 is the set for which $x+y=7$, this set has no intersection with the set for which $x+y=5$, no matter what kinds of numbers x and y are. Why? Simultaneous inequalities with similar restrictions could be considered as well.

What is a method of finding the intersection of two sets of number pairs? Consider

F_2: (x, y) where x and y are rational numbers for which $x+y=5$;

F_9: (x, y) where x and y are rational numbers for which $x+3y=13$.

We want a pair or pairs (a, b) that satisfy both. This may be thought of as seeking two sets: one for which $x=a$ and one for which $y=b$ that have the same points in common as F_2 and F_9. Geometrically this means that we are seeking a horizontal and

a vertical line which pass through the intersections of F_2 and F_9.

How can we represent the set of lines which pass through the intersection of F_2 and F_9? We may show that

$$(1) \qquad (x+y-5)+c(x+3y-13)=0$$

defines, for each constant c, a set of points whose graph is a line through the intersection of the graphs of F_2 and F_9. We then want to choose a c so that the graph of (1) is a horizontal line, i.e., so that (1) is free of x. This may be done by taking $c=-1$ which gives $y=4$. Similarly one may eliminate y by taking $c=-\frac{1}{3}$. Actually there would be more symmetry of treatment if (1) were replaced by $d(x+y-5)+c(x+3y-13)=0$ and c and d chosen correspondingly.

Similarly, if we have F_2 and F_3 above, we might consider

$$(2) \qquad (x+y-5)+c(4x^2-y)=0.$$

Here $c=1$ eliminates y and gives $4x^2+x-5=0$, i.e., $x=1$ or $x=-5/4$. We cannot choose a constant c that will eliminate x, but we can find the intersection of $x=1$ with $x+y-5=0$ that must pass through the intersection of F_2 and F_3. The y turns out to be 4. Similarly it may be found that when $x=-5/4$, $y=25/4$. Although $(-5/4, 25/4)$ belongs to the set of ordered pairs which is the solution set of the two equations $x+y=5$ and $4x^2=y$, it is not an element of the intersection of sets F_2 and F_3. The intersection set of these two sets is $\{(1, 4)\}$.

Other new topics: Directed numbers correspond to vectors on a number line; complex numbers correspond to vectors on a plane. Vectors are also appropriate in geometry; trigonometry may well be introduced in connection with many different topics, including the geometry of the sphere.

Elementary notions of statistics are appropriate for junior high school; inferential statistics and probability are suitable for the twelfth grade.

All students should receive an opportunity for contact with some of the newer mathematical approaches. (Certainly the old approaches are not working too well for below-average students.) *We do not yet have enough experience to say where each topic and idea should be introduced nor even just what topics are most appropriate.* This is a problem which calls for careful study and experimentation. A teacher experienced with these topics and ideas will be able to capitalize on opportunities as they occur and he should be free to do so. It is more important that fundamental ideas and basic techniques be well understood than that high levels of advanced computational ability be developed at the secondary level.

Topics that could be eliminated are extensive and tricky factoring, simultaneous quadratics, involved and spurious fractions, most of business mathematics, extensive computational problems in algebra and geometry. Word problems are useful for their training in translation to and from mathematical symbolism, but there is no point in training in techniques for solving by type. Though some logarithmic computation should be retained so that students may feel at home with it, involved solutions of triangles should be reduced. Logarithms are most important for their connection with exponents and, therefore, are primarily algebraic in nature. Formal synthetic proof in geometry can easily be overdone. Though a student should be able to produce on demand a reason for each statement, endless repetition of, for instance, "two things equal to the same thing are equal to each other" or "side-angle-side" are out of place. Also, wherever other tools, like algebra, make a proof easier, they should be considered.

The treatment of topics, new and old, in grades seven, eight, and nine is considered in the next section, and in geometry, in the following section. At all levels of instruction and in all areas of subject matter such treatment must be entirely within the context of the background of both teacher and pupil.

Mathematics for grades seven, eight, and nine. The mathematics program for these three grades has given concern for a good many years. Many believe that the mathematics curriculum, in grades seven and eight especially, is inadequate to meet modern needs. It is often stated that lack of interest in mathematics and science in senior high school and college may be due largely to unsatisfying experiences in the junior high school.

In grades seven and eight much of the time is given to so-called social applications, some of which seem inappropriate at this level. Few new mathematical ideas are introduced. Teachers often report that the traditional courses at this level offer little challenge to the upper 50 per cent of the pupils, and that review and maintenance provided for the lower 50 per cent usually serve only to deaden their interest because emphasis is usually on drill, with little provision for bringing out deeper insight and understanding. New programs in mathematics for these grades seek to give basic instruction for all in the fundamental skills, concepts, and principles of arithmetic, supplemented by significant topics from algebra and geometry. Provision for upper 50 per cent is sought through enrichment of understanding as well as acceleration of accomplishment.

Studies by Piaget support the belief that boys and girls twelve years of age are able to work with mathematical ideas at a considerably higher level of abstraction than is characteristic of traditional mathematics courses in grades seven and eight in this country. A recent study at the University of Wisconsin indicates that basic principles of number relations and operations and use of "if-then" statements (as traditionally taught in plane geometry of grade ten) were taught successfully, with some degree of abstraction, to eighth-grade pupils involved in the investigation.

In 1958 the Commission on Mathematics of the College Entrance Examination Board published a small brochure, *The Mathematics of the Seventh and Eighth Grades*, in which is listed what the Commission believes to be essential in the foundation for the Commission's high school program for all college-capable students. This list is probably the best current statement of appropriate content for the mathematics of grades seven and eight.

As a part of the University of Maryland Mathematics Project (Junior High School), UMMaP, started in the fall of 1957, forty-three teachers, most of whom are in the Washington, D. C., area, have been teaching an experimental seventh-grade course during the school year 1958–59. As a result of this experience the seventh-grade course will be revised and an eighth-grade course will be prepared for use in these same and other schools. It should be understood that this is an "experiment" and that one of the major concerns is to try to determine if certain concepts *can* be successfully taught at this grade level.

During the summer of 1958 the School Mathematics Study Group (SMSG) began its program by convening in writing session a group of forty research mathematicians and teachers of secondary mathematics. As a part of their program some fourteen "experimental" units were prepared for tryout in grades seven and eight in twelve educational centers across the United States. These tryouts are considered essential to the preparation of sample textbooks for these grades. It is reasonable to expect that these sample texts will contribute materially to the structure of the mathematics program of grades seven and eight, and that the concepts to be developed in these texts will influence strongly the development of the junior-high-school mathematics curriculum of the future.

Units on such topics as nondecimal numeration, factoring and primes, nonmetric geometry, averages, finite systems, approximations, and chance are included in the SMSG experimental materials for grades seven and eight and in the UMMaP

seventh-grade course. Neither group, at this time, proposes that these topics *should* be included in the courses for these grades, but these concepts are among those mentioned as important in the total mathematics curriculum by all, or nearly all, of the current curriculum study groups.

The Commission on Mathematics has made detailed recommendations recently on the mathematics of grade nine for college-capable students. These recommendations should have a profound influence on the traditional first-year algebra course, and all junior high school mathematics teachers are urged to give them careful study. The work of the University of Illinois Committee on School Mathematics has received wide recognition. The course for grade nine has been tried out in quite a number of schools and has served as the basis of in-service study for many teachers.

There are some who recommend that the current, rather widespread, practice of offering both general mathematics and algebra in grade nine should be discouraged in favor of teaching the regular first-year algebra more slowly for the slower group and providing opportunity for individual excursions into deeper understanding for the gifted. There are others who feel that such a program would in no sense make proper curricular adjustments for the slow-learning pupil. There are still others who argue that first-year algebra should be placed in the eighth grade with a corresponding step-up in the program of subsequent grades. These facts lend support to the statement that, at this level of instruction, the adjustment of subject matter and teaching techniques to varying levels of pupil ability is a problem which demands careful study and experimentation.

How geometry should be introduced and developed. Those who teach geometry in our schools should be assured that they are making a vital contribution to the education of their pupils. Synthetic plane geometry taught from a suitable text by a competent teacher provides magnificent opportunities for bringing to life in the minds of the pupils latent powers of analysis, insight, and reasoning. No other mathematics course contains such a wealth of nontrivial original exercises of all degrees of difficulty. Pupils have the experience first of proving fairly simple (almost immediate) consequences of various theorems, and later of solving more sophisticated originals based on a more miscellaneous set of theorems. The solving of such originals will involve considerable thought, trial-and-error experience, and a gradual development of an awareness of essential needs. When a pupil has, after considerable effort, solved any original of the more challenging type, he has gained a great victory. The experience can make a significant contribution toward the increase of his power over any situation, mathematical or otherwise, requiring sustained and creative thinking.

It is the considered opinion of this committee that elements of geometry should be taught throughout the secondary sequence (grades seven to twelve) and that one year should be devoted to synthetic geometry with particular emphasis on the geometry of the plane. This does not imply any desire to minimize the importance of co-ordinate geometry. Indeed, the study of co-ordinate systems should permeate the entire secondary sequence, beginning with the idea of a one-to-one correspondence between numbers and points on a line in grade seven. The year devoted to synthetic geometry should be no exception. For example, it is instructive to extend the study of loci to the co-ordinate plane. The student can profitably consider sets of points whose co-ordinates (x, y) satisfy such conditions as: $2x+3y=7$; $x^2+y^2>25$; $y>x^2-x-6$; $|x-y|<7$, etc. It is also desirable to use analytic methods, on occasion, to provide alternative proofs for certain theorems which have been proved synthetically. Such proofs give pupils confidence in their ability to use analytic methods.

During the year course some teachers with superior classes may find it possible to include a considerable amount of solid geometry along with the plane geometry without sacrificing the essential values of the latter. Indeed, some references to space geometry do much to enhance the pupils' understanding of relations that were first considered in the plane. The first consideration should be the presentation of a thorough course in plane geometry in which pupils receive intensive training in the solution of a large number of miscellaneous problems. In this situation the pupil needs time for contemplation, time for the formulation of conjectures, and time to test these conjectures for consistency with known facts. On the other hand, there is no point in pursuing an endless elaboration of originals that are confined to the plane. At some point during the year the good student will find it both natural and profitable to consider solid geometry in a systematic manner and not merely as counterpoint for plane geometry.

The amount of solid geometry to be included must be determined by the teacher. It will depend, among other things, on the ability of the class and on the nature and extent of the instruction given in previous grades. In any case, no effort should be made to cover a predetermined amount of solid geometry if it becomes apparent that the time available is insufficient for a given class. Such forced acceleration is deleterious and completely foreign to the spirit which should prevail in a mathematics classroom. Moreover, the necessary elements of solid geometry can be woven into other parts of the curriculum.

After a careful analysis of the content of solid geometry, this committee is inclined to agree with the widely held view that it is neither necessary nor desirable to devote a full semester to deductive solid geometry. It is not necessary because most of the area and volume formulas for prisms, pyramids, cylinders, cones, and spheres are considered first in grades seven and eight, and repeatedly applied in grades nine and ten. This should be true also for such basic concepts as those pertaining to lines and planes in space; dihedral, trihedral, and polyhedral angles; and polyhedral forms. Such concepts as these now are presented, usually for the first time, at the beginning of the semester course in solid geometry.

These concepts should be used as a basis for exercises in drawing in grades seven and eight, and the understanding thus acquired should be maintained and extended by suitably chosen problems requiring drawing and three-dimensional analysis in grades nine through twelve. Some of the material on the geometry of the sphere can be presented in connection with certain topics in trigonometry if it has not been considered earlier.

The conclusion that a full semester of deductive solid geometry is not desirable rests on two opinions, which represent the consensus of this committee. First, solid geometry is not a good place to study deductive proof. Many of the proofs tend to be long and cumbersome and their presentation does little to enhance the understanding of deductive methods that the pupil gained in plane geometry. Second, on the basis of relative values, deductive solid geometry is not worth a full semester of the pupil's time.

A college-preparatory program is presented below for grades nine through twelve. The principal feature of this program is that it provides opportunity for a varied program in grade twelve by absorbing solid geometry into other parts of the program.

Ninth grade—algebra
Tenth grade—geometry (plane with some solid)
Eleventh grade—algebra, trigonometry
Twelfth grade—any two of the following semester courses: probability and statistics; analytic geometry; mathematical analysis based on a study of functions (algebraic, trigonometric, exponential, and logarithmic). Some schools might find desirable a strong course in analytic

geometry and calculus as preparation for the Advanced Placement examinations.

The view that the elements of geometry should be taught throughout the secondary sequence requires fuller explanation. Before we consider what elements should be taught at each level we wish to call attention to two principles which we believe to be valid.

1. The learning experiences in geometry that precede the one-year course should be *valuable in themselves* and not solely as prerequisites for the deductive geometry that is to follow. We must remember that we are concerned with the education of all youth, not merely those who are college-capable. This consideration is particularly important in grades seven and eight. In these grades there must be provided a program of instruction that is appropriate for pupils whose future study of geometry must be confined largely to its applications, as well as one that is appropriate for those who will later study geometry as a deductive system.
2. Those learning experiences that are designed as preparation for deductive geometry should not require any great amount of verbalization of theorems that are to be proved later in the formal course. The time can be employed to better advantage in developing certain concepts that provide a foundation for the subsequent study of deductive proof. In cases where it is necessary to deal with statements pertaining to congruence, similarity, relations between the sides of a right triangle, etc., there should be constant emphasis on the difference between proof and experimental verification.

The following elements of geometry are regarded as appropriate at the various grade levels:

Grades seven and eight—(the introduction or observation stage):

- Drawing, both freehand and with straightedge and compass
- Drawings that represent three-dimensional objects
- Perspective drawing
- Conjectures based on inductive and intuitive reasoning
- Use of measuring instruments, ruler and protractor
- Recognition of the fact that all measures are approximate
- Relative error, distinction between precision and accuracy
- Experiments designed to test conjectures about the areas and volumes of certain solids
- Construction of templates for some solids

Concepts: Application of set language to geometry; the set of points on a line, the set of lines on a point, the set of lines on a plane, the set of planes on a line, intersection of two sets; one-to-one correspondence; simple locus problems in two and three dimensions; congruence, ratio, similarity, concurrence of lines, collinearity of points; "inside" and "outside" of a closed curve; the "if-then" form of a statement; exercises which are "open-ended" in the sense that the pupil is given a certain set of data or a certain drawing and asked to draw as many conclusions as he can.

The above material might well constitute from 20 to 30 per cent of that presented in grades seven and eight.

Grade nine:

Liberal use should be made of geometric drawing in illustrating problems wherever possible. The one-to-one correspondence idea should be applied to the number scale and to the development and use of the Cartesian co-ordinate system.

Grade ten:

Current textbooks in plane geometry present the following topics with some variations in order: Fundamental ideas (terminology, drawing, introduction to deductive proof); triangles; parallel and perpendicular lines; polygons including the work on areas and a proof of the Pythagorean theorem; circles, similar polygons, loci, and regular polygons with an informal argument that develops the formulas for the circumference and area of a circle as limiting cases. When we consider this sequence in terms of what has gone before, it seems clear that some time should be saved, not because some of the work has been done earlier (although this may be true in the early units), but because a firm foundation for understanding has been provided.

Grades eleven and twelve:

In these grades the pupil's knowledge of coordinate geometry should be reviewed and extended. This extension may include some work with co-ordinate systems in three dimensions. The pupil's ability to draw and his knowledge of geometric principles should be used to provide geometric interpretations of a wide variety of problems. In particular, the trigonometry should include some consideration of three-dimensional situations that require the pupil to use both the drawing techniques and the theorems which he learned in geometry. Students who study calculus need experience of this kind.

Attempts will be made by various groups to improve the content and organization of the year course in geometry and

to provide a better postulational base for the theorems it contains.[2]

We wish to encourage all such attempts and we shall try to evaluate and interpret these proposals when they become available. In this report however, we are concerned with what can be done to improve geometry *now* when the results of these attempts are not yet widely accessible, to say nothing of being widely accepted.

We are of the opinion that the sequence for grade ten described above provides an adequate basis for the study of deductive proof. We should strive for improvements in the teaching of school geometry that can be achieved within the framework of the present program as well as for those which require rather fundamental modifications of the program. Improvements which require little or no modification of content include:

1. The use of the wonderfully expressive language of sets in the development of certain geometric concepts. This development should be an extension of that used in grades seven, eight, and nine.
2. A sharper delineation of the meaning of deductive proof. This requires a review and extension of some of the basic ideas involved in if-then reasoning. The pupil should be thoroughly familiar with the inverse, converse, and contrapositive forms of a simple theorem[3] and should recognize the logical equivalence which exists between a theorem and its contrapositive. Some superior students may learn to apply these ideas to theorems that have more than one condition in the hypothesis.
3. Better understanding of the nature of an indirect proof. This in turn requires an understanding of the meaning of negation. Pupils should become confident of their ability to establish a conclusion by showing that its denial leads to a contradiction.
4. More study of miscellaneous lists of problems. Time should be provided toward the end of the plane geometry sequence for the solution of problems of "rare charm and distinction." Such problems are plentiful and lists of them should be provided which are unclassified with respect to the theorems required for solution.

   For the best pupils supplementary lists of exercises can be provided that require the understanding and application of several of the more advanced theorems dealing with concurrent lines, collinear points, and cyclic quadrilaterals.

   The kind of thinking that pupils are called upon to do in solving geometric originals is somewhat like that required of the mature classical mathematician. They must call upon their store of mathematical information and make use of all the intuitive powers at their command. It is important to realize that the pupils' intuitive powers must be developed as well as their ability to manipulate symbols in accordance with prescribed rules. The study of geometry stimulates the use of the imagination in the study of drawn figures, and develops the knack of using only those properties of these figures specified by the chosen hypotheses. This kind of perception is important not only to mathematicians but also to scientists and engineers.
5. Better understanding of geometry as a postulational system. Students should be made aware of the fact that each theorem can be traced back to the fundamental assumptions (axioms and postulates) that were made in the beginning. They should occasionally consider "families of theorems" that the text presents in a certain sequence, and should inquire if it is logically feasible to arrange these related theorems in different sequences. In addition, pupils should encounter a few of the startling consequences which result from a change in the postulational base. For this reason, there should be some consideration of the geometry of the sphere and some reference to other non-Euclidean geometries. The postulational method cannot be fully appreciated on the basis of just one example, particularly when that example produces results which are, at the beginning at least, intuitionally evident. The knowledge that other systems exist in which logically derived conclusions defy intuition is an important element in the cultural heritage one should acquire from the study of geometry.
6. More adequate provisions for individual differences in ability to learn geometry can be made. This committee is concerned with curricular provisions for all pupils, not merely those who are college-capable. Provisions advocated later in this report imply that, whenever possible, sections should be provided which allow for presentations which vary widely with respect to the time spent on various topics, the level of abstractions expected, the degree of penetrations required, and the amount and kind of problem materials used.

   For able pupils geometry should be "a study of the ways in which new ideas are formulated and combined, of the ways in

[2] In this connection we should consider the current work of the School Mathematics Study Group, the University of Illinois Committee on School Mathematics, the Ball State Group under C. F. Brumfiel, and the Commission on Mathematics of the College Entrance Examination Board, among others.

[3] A "simple theorem" is here defined as one that has one condition in the hypothesis and one statement in the conclusion. For example, if $a=b$, then $3a=3b$.

which new ideas follow in logical order from ideas already formulated, of the reasons why the ideas developed follow one another in that order and so on."[4] Pupils who are less able to deal with logical abstractions will need correspondingly greater emphasis on the content of geometry as a body of information about lines and angles, surfaces and solids, etc., together with a large variety of practical applications of geometry, before they can begin to appreciate the logical structure of the subject. Some of these pupils may succeed better if their study of geometry is postponed to grade eleven.

It is evident that this provision of a variety of courses is likely to produce some courses which do not meet the requirements of a college-preparatory sequence in mathematics. A course that presents only the facts of geometry and pays little or no attention to the essentials of deductive proof certainly does not meet these requirements. While such courses may provide valuable learning experiences for some pupils, they should not be described as deductive geometry. This description should be reserved for those courses which are designed to advance the pupils' understanding of the nature of proof.

The importance of geometry in the school curriculum serves to emphasize the need for better training for those who expect to teach geometry in grades seven to twelve. Some of the criticism of geometry in our schools is found, on analysis, to stem from inept teaching rather than from deficiencies in the content or organization of the course. This is not surprising when we consider the fact that many teachers of geometry *have not had a course in geometry since their high-school days.* This situation must be remedied. The remedy is not to be found in requiring prospective teachers to take courses in geometry that are primarily designed for research-minded people who seek advanced degrees in mathematics. Instead, courses should be developed which give prospective teachers a deeper understanding of the ideas we encounter in school geometry, that is, courses that consider these ideas from an advanced point of view. If the teacher of mathematics is to convey to his pupils some understanding of the essentials of logical reasoning and the nature of proof, he must have some training in these concepts himself.

[4] E. P. Northrop, *Fundamental Mathematics* (Chicago: University of Chicago Press, 1944), Vol. I.

Implementation of the curriculum[5]

Provisions for individual differences. It is recognized that the ability to learn varies greatly among all human beings, and that the variation in this ability to learn is as great in mathematics as in other areas of knowledge. By the time pupils enter the secondary school at the seventh grade there is a wide range of achievement in mathematics from the very able pupils to the very slow pupils. Thus the secondary school must make instructional provisions for this variation in achievement so that pupils may proceed successfully and at the rate adapted to their several abilities.

The question arises as to what mathematics the pupils should study. The answer appears to be: the same mathematical structure and concepts, but varying in amount, in complexity, in depth, and in manner of organization and presentation in order to be consistent with the pupil's ability and his past achievement. All pupils may eventually study the same algebra, but not at the same time and at the same rate, to the same depth of understanding, with the same application, or necessarily in the same sequence. Thus, for example, in the ninth school-year one group may be doing a development of the number systems from an axiomatic point of view, another group may be studying positive and negative numbers as usually presented in an inductive manner, while a third, less able, group may be making use of the basic concepts of algebra to help

[5] A very helpful pamphlet on "Teaching Aids," prepared by a subcommittee of the Secondary-School Curriculum Committee, has been published by the National Council of Teachers of Mathematics. Emil Berger and Donovan A. Johnson were co-chairmen of this subcommittee. The pamphlet is available through the office of the Executive Secretary, 1201 Sixteenth St., N.W., Washington, D. C.

strengthen their understanding of arithmetic processes.

A sequentially structured study of high-school mathematics is rather easy to devise for the more capable pupils. The learning of this same structure must be modified for the average group of pupils. For the slow pupils the learning may have to be fragmented and accompanied with much work in applications and examples. The essential point is: If the mathematics is to have value, it must be ultimately a body of concepts and understandings built into a related system recognized not only as a significant part of our culture but also as providing the individual with a meaningful tool either for solving his problems or for continuing his education in more advanced study of mathematics.

The mathematics program for the slow learner thus contains the same basic structure, the same usefulness, and many of the same concepts as it does for the average or fast learner. However, the teaching methods by which the slow learner comes to grasp this structure of knowledge will undoubtedly be different from that of the other learners. For example:

1. Generalizations, in order to be understood by the class, must be preceded by many and varied concrete illustrations.
2. Frequent reviews in meaningful situations are necessary in order to maintain a reasonable level of skill and understanding.
3. Laboratory techniques and manipulative devices should be used freely.

Furthermore, it is axiomatic that great precaution must be taken not to make the program for the slow learner one that consists of a series of completely unrelated topics, meaningless tricks, and useless applications, or one of mere amusement and entertainment whose primary purpose is to keep the pupil in school rather than to provide him with appropriate opportunity to acquire pertinent mathematical information.

For pupils of below-average ability the principal emphasis should be upon basic understandings of a carefully considered program of minimum essentials. Until a better definition of a minimum program in mathematics has been determined for the high school, the Check List of Functional Competence prepared by the Commission on Post War Plans[6] can continue to serve as a satisfactory list. Opportunity should be sought constantly for extending those who are capable beyond such a minimum program.

A carefully planned program for motivation of interest and direction of effort is necessary. The well-prepared teacher is also essential. Particularly for the below-average group, teacher attitude is extremely important. Specific care should be taken to prevent stigmas being attached to the program. Each teacher must be willing to accept the pupils as they are and to help them feel secure and wanted. The teacher must have an enthusiasm for the work as well as imagination and understanding in dealing with pupils. Supplementary materials can be very helpful and effective.

A sample of topics which can be adapted to provide motivation and help build prestige in such a program might include the following:

Elementary work with other number bases
Ancient methods of computation: Russian peasant method, finger multiplication, galley method of multiplication
Indirect measurement, including construction and use of a clinometer and hypsometer
Other systems of numeration
Magic Squares
Experimental geometry
Use of the slide rule
Probability and statistics; an experimental laboratory approach.

As guides in formulating the most effective program possible the following principles can be helpful:

1. A four-year sequence should be made available to *all* pupils in grades nine through twelve.
2. Two years of mathematics should be required for graduation from all secondary schools.
3. Where enrollment warrants, below-average pupils should be placed in separate classes, limited in size to a maximum of twenty pupils.

[6] Commission on Post War Plans, Guidance Report, The Mathematics Teacher, XL (1947), 318–19.

4. It is the responsibility of the teacher to locate deficiencies and provide for remedial instruction within the classroom.
5. If, at any time, a pupil shows sufficient growth and promise it should be possible for him to move into the program for more able pupils.

In each of the four years, provision should be made for the maintenance and extension of proficiency in arithmetic. Basic algebraic concepts should be developed, especially those that may be used to help develop an understanding of arithmetic processes. Emphasis should be placed on geometry and the pupil given opportunities to make discoveries and form generalizations based upon laboratory experience. Attention should be given to the notions of measurement. By means of laboratory experiences the basic concepts of probability and statistics should be developed. In later years, community resources could well be used to develop certain notions of consumer mathematics.

Much research is needed to provide teachers with a detailed outline, including specific resource units, for a four-year course of study in mathematics for below-average pupils.

Probably the greatest challenge to teachers of mathematics from the point of view of potential benefit to the welfare of our society lies in the recognition of, and the making provision for, the mathematically gifted pupil. It is not likely that any serious thinker would take much exception to this statement from a leading educator: "The future of our country and of democracy as a way of life depend to a considerable degree upon the widespread recognition and development of our greatest resource—gifted children and youth."[7] What then are the characteristics of the mathematically gifted, and how can we provide a program of significant challenge and interest for them?

The recognition of the mathematically gifted pupil can be accomplished most effectively through the identification of certain basic characteristics, some general and others more specifically mathematical in nature.

1. General characteristics
   (a) Makes associations readily and retains them indefinitely
   (b) Recognizes similarities and differences quickly
   (c) Has excellent memory, good vocabulary, broad attention span, and high reading ability
   (d) Has a relatively mature sense of values
   (e) Pursues interests with tremendous energy and drive
   (f) Uses his spare time productively

2. Special characteristics
   (a) Recognizes patterns readily and enjoys speculating on generalizations
   (b) Prefers to think on higher levels of abstraction
   (c) Classifies particular cases as special cases of more general situations with relative ease
   (d) Follows a long chain of reasoning, frequently anticipating and contributing
   (e) Frequently asks profound questions
   (f) May be reading mathematics books years ahead of his class
   (g) Is frequently impatient with drill and details that he thinks are not important

All schools should provide opportunities for the gifted. Ability grouping is a widely acceptable and desirable practice, and special classes for the able are recommended where organization and staff facilities permit. Honors classes, seminars, and special projects provide excellent means for challenging the gifted. A policy of acceleration is followed in some areas. Two basically different patterns are used: (1) a narrow type, or "mere speed-up" plan, in which pupils are allowed to "cover" the requirements of one year's program to get into that of the next year—for example, two years of algebra in one year; and (2) a broad type, or "faster-growth-in depth" plan, where more mature understandings and procedures are expected sooner.

There is some question about the advisability of the "mere speed-up plan." A sophomore pupil, for example, who is undertaking senior mathematics could easily be a triple loser. He could be so completely

[7] Paul H. Witty, "Today's Schools Can Do Much for the Gifted Child," *The Nation's Schools*, LVII (1956), No. 2, 72.

out of step with the educational pattern that he would be confronted with unnecessary confusion in planning for the future; secondly, he could have lost material benefits by not having had the opportunity to make significant explorations below the surface of mere satisfactory accomplishment; and thirdly, he could have missed the inspiring challenge that can come from inquisitive search into the rationale of procedures, the foundations of concepts, and the implications of principles. On the other hand, a carefully planned acceleration program of the "faster-growth-in-depth" type can benefit the able pupil very materially through the provision of opportunities for deeper investigation of foundations, broader contact with related areas of cultural and utilitarian value, and more significant appreciation of basic interrelationships as well as advancement in grade-level accomplishment.

Any program of enrichment for the mathematically gifted not only should provide real challenge to excellent accomplishment but also should provide basic information essential to top-level performance in mathematics, both for the present and for the future. This can be accomplished in better fashion by less emphasis on drill and on delineation of applications, in order to provide more opportunity for discovery of significant patterns and formulation of generalizations. Under informed guidance such pupils should be encouraged to read widely, to investigate significant topics which offer them challenge, to participate in competitive endeavors, and to seek occasion to meet with other mathematically gifted pupils from their own school and from other schools.

Since the purpose of ability grouping in mathematics is to increase the efficiency of learning, it behooves teachers to adjust their methods of teaching in order to take maximum advantage of the homogeneity of the several groups and to gear instruction to the learning level of the students in each group. Furthermore, administrators should strive (1) to develop in their staffs an understanding of the basic purposes of ability grouping and the desirability for taking advantage of it, (2) to adjust class size for optimum benefits from such grouping, and (3) to reduce nonteaching duties so that all teachers will have better opportunity for taking care of individual differences at all levels of instruction.

The organization of the program. Whatever pattern the revised curriculum in mathematics might assume, it should be characterized both by uniformity and flexibility. While the program should be sufficiently uniform to meet national needs in minimizing problems of transfer from one school system to another and problems of personnel, it also must be sufficiently flexible to meet both regional and local demands. This curriculum must reflect a total program in mathematics, the instructional responsibility of which will be (1) to guarantee an appropriate minimum program for every educable individual whether slow learner, average pupil, or mathematically gifted, and (2) to provide a sufficient program to challenge the interest and ability of the most able. The educational responsibility of each grade level in this total program must be appraised very carefully. It can be stated now, however, that at any grade level the principal responsibility of the teacher of mathematics is to do the very best job of which he is capable in helping his pupils acquire fundamental understandings of the basic concepts, principles, and techniques of mathematics. Applications of mathematics are important as media through which pupils might gain deeper appreciation of the tool value of mathematics and as aids in the clarification and illustration of mathematical content. They should not, however, be used at any grade level as if they constituted the basic significance of the mathematics curriculum. Technology is subject to rapid change. Training in specifics can, and may, soon become obsolete. On the other hand, a person with fundamental training in mathematics will

have the background for making adaptations to applications, even to those not now foreseen.

Teachers are asking many questions concerning content and grade placement of mathematical subject matter. A few representative questions are: What about the relative placement of plane geometry and a second course in algebra? Should the first course in algebra be placed in the eighth grade? Should the high school program treat of probability and statistics?

These are difficult questions, but current thinking seems to indicate fairly clear-cut answers. As the courses in algebra and plane geometry are now organized, it seems to make little difference which immediately follows elementary algebra. In the revised curriculum of the near future the new emphases on algebra and geometry will more than likely remove the necessity of this question. At present it seems unwise to make a blanket recommendation for the placement of algebra in the eighth grade. It is conceivable, however, that there are situations in which it is desirable to follow such a program, in which case the responsibility of the ninth grade will be a follow-up program in algebra. In such situations the pupils should be mathematically superior and they should be taught in homogeneous groups by well-qualified teachers. The wise use and interpretation of statistical data is becoming more and more important in the daily life of man. Some elements of statistics are already incorporated in the current programs of grades one to twelve. Examples are: numerical representation of quantities, averages, and graphs. In the light of the increased emphasis on statistics in this technical era it is quite likely that these and other statistical areas may demand more careful attention in these grades. However, the study of inferential statistics probably should be postponed until grade eleven or twelve.

In view of the many different types of school organizations, i.e., 8-4, 6-3-3, 6-6, there are definite problems of articulation between schools. These can be surmounted, provided the same certification is required of teachers at a given grade level regardless of the type of school organization.

The demands for mathematics are becoming greater; therefore those students who desire to do so should have the opportunity to study mathematics during each year of their education. Many schools are concerned about the content of the twelfth year of mathematics. The present thinking of the groups considering this problem is that the twelfth year should provide some choice of advanced courses, when possible, to provide for varying characteristics of interest, aptitude, and need. Such provisions should be made in schools only when a class of mathematically superior students can be found and a teacher is available who has taken mathematics beyond elementary calculus. Furthermore, for the students who have not reached the degree of mathematical maturity necessary for such courses as those implied above, provision should be made in the twelfth year for them to take work in intermediate algebra, trigonometry, or certain types of applied mathematics.

Every encouragement should be offered to able pupils tocontinue in advanced high-school mathematics even though immediate or even ultimate need is not evident. After all, one cannot use what he does not know. One guiding principle in requiring certain mathematics courses is that effort should be made to provide pupils with study opportunity and accompanying professional guidance so that later they may not be denied access to higher levels of education because of a deficiency in background preparation. This may mean that more mathematics will be required of those able to master it.

There is a minimum amount of mathematical competency desirable for all high school graduates. There are some students whose future plans will not require more than this minimum. There should always be some method to determine whether or not minimum proficiency has been at-

tained. Probably the best place in the program to include such a measure would be the eleventh grade. Provision for those pupils who desire to continue beyond the minimum program is the joint responsibility of the college and secondary school.

Administration and supervision of the program.[8] No program, regardless of how carefully and thoughtfully it might be organized, can be effective unless it is supported and encouraged by a sympathetic and interested administrative attitude. One of the major problems calling for close co-operation between teacher and administrator is that of making appropriate provision for taking care of individual differences. Within classes, ability grouping makes such provision, but this practice needs to be supplemented, wherever possible, by multitrack programs providing differentiated sequences running through the several grade levels. In general, the number of tracks will depend upon the size and needs of each particular school. The basic differentiation between tracks will be in terms of variations in academic objectives. Usually at least one of the tracks will be an accelerated program. It is also very important that flexibility be recognized as a significant characteristic of the multitrack program. For example, pupils who are especially gifted in mathematics, or who are able and sincerely interested in mathematics, should be encouraged to take top-track classes in mathematics but should not necessarily be expected to take top-track classes in other areas.

Although there has been a desirable trend toward consolidation, there still remain many high schools so small as to be unable to provide for homogeneous groups and multitrack programs. Some of the ways by which such schools can provide for the varied needs of its pupils are: offering correspondence courses for gifted pupils; rotating offerings; having a master teacher co-ordinate the work of gifted pupils by bringing in mathematicians and scientists to speak and confer with them; providing information about colleges; arranging for a traveling library; using films and other visual aids; presenting special guidance and counseling; and setting up special projects for individual pupils.

Whether in a small school or a large school, mathematics courses should require work in addition to that done in class. This should be planned as a meaningful supplement to the work done in class, designed to increase the pupils' mastery and competence, and not as mere busy work to keep pupils occupied. All out-of-class work should be differentiated according to the needs and abilities of the pupils. Any such work intended for honors credit should be offered on an optional basis, but capable pupils should be given strong encouragement to take advantage of such opportunities. The standards of achievement should be higher for honors courses and, in general, for all elective courses than for required courses. The administrative policy of all schools should require that high standards of achievement, respectably rigid and reasonably attainable, be maintained in all courses. Furthermore, there should be a positive program for the creation of a school atmosphere of open respect for academic achievement. Some ways in which this may be accomplished are: recognize specific cases of such achievement by such means as awards, assembly programs, banquets, and clubs; provide the press with information about scholarship; encourage participation in local and national contests.

One of the major responsibilities of the administrative staff of any school is to provide for the pupils an intelligent program of counseling and guidance. This is just as important within subject areas as in general psychological and vocational counseling fields. For example, no pupil should be

[8] See the pamphlet prepared by the Subcommittee on Administration of the Mathematics Program, *The Supervisor of Mathematics, his Role in the Improvement of Mathematics Instruction* (Washington, D. C.: National Council of Teachers of Mathematics, 1959). Veryl Schult was chairman of this subcommittee.

allowed to enroll in an elective course or a sequential course if there is substantial evidence that he does not have the prerequisites necessary for success in the course. On the other hand, all pupils should be encouraged to enter the courses of greatest challenge to them and to strive for the highest level of accomplishment of which they are capable. Informed counseling should be based on a comprehensive pattern rather than on discrete bits of information about each pupil. Capable pupils should be guided into participation in such programs as the Advanced Placement Program of the College Entrance Examination Board, the Merit Scholarship Program, and other programs based on excellence of achievement. Positive efforts should be made to recognize the college-capable pupil early so that he may be directed into a program of study which will equip him for college entrance and for a successful college program. The administrative staff and the teachers should keep themselves informed about the activities of agencies which affect the curriculum of college-capable students, such as the Commission on Mathematics of the College Entrance Examination Board, the School Mathematics Study Group, and the Committee on the Undergraduate Program of the Mathematical Association of America.

While testing is not a panacea for all the ills of mathematics instruction, the judicious use of school-wide, city-wide, or county-wide tests can be productive of very helpful results. Such a program should use both teacher-made and standardized tests[9] and should be under close administrative supervision to prevent overtesting and improper use of test results. Tests are usually designed for rather specific purposes, and a test planned for one use may or may not be appropriate for another. Some of the purposes for which tests may be used are: aid in determining levels of pupil achievement; aid in determining areas of pupil weakness; aid in making decisions about vocational and academic futures of pupils; aid in identifying slow pupils and mathematically gifted pupils; aid in establishing standards of achievement; and aid in determining the effectiveness of a specific curriculum. The written test is not the only means of effective evaluation. Other sources for gathering pertinent information about the achievement and progress of pupils are oral tests, interviews, anecdotal records, observation, informal conversations, pupil projects, and various other forms of out-of-class work.

Regardless of how complete the evaluation program may be, it can still lose the major portion of its true significance if teachers and administrators fail to use proper procedures and limitations in the interpretation and use of the results. Such personnel need to be informed as to the differentiation in purposes of different types of tests, the mechanics of fundamental statistical procedures and the principles of interpreting the results they produce, and the distinctive characteristics of various testing techniques.

The teacher. The most important single factor contributing to the effectiveness of any program of instruction is the teacher. As in any true profession, the competent teacher is characterized by scholarship in relevant knowledge and proficiency in the use of efficient techniques. In view of current curriculum demands teachers of mathematics in grades seven through twelve will need to have competence in (1) analysis —trigonometry, plane and solid analytic geometry, and calculus; (2) foundations of mathematics—theory of sets, mathematical or symbolic logic, postulational systems, real and complex number systems; (3) algebra—matrices and determinants, theory of numbers, theory of equations, and structure of algebra; (4) geometry—Euclidean and non-Euclidean, metric and

[9] A pamphlet of such tests in mathematics, prepared by Sheldon S. Meyers and sponsored by the Secondary-School Curriculum Committee, is available through the office of the Executive Secretary of the National Council of Teachers of Mathematics, 1201 Sixteenth St., N.W., Washington, D. C.

projective, synthetic and analytic; (5) statistics—probability and statistical inference; (6) applications—mechanics, theory of games, linear programming, and operations research.

Ideally, every teacher of secondary mathematics should have completed successfully a five-year program, emphasizing the above areas and culminating in the master's degree. As a minimum, teachers of mathematics at the seventh- and eighth-grade levels should have completed successfully a program of at least eighteen semester hours, including six semester hours of calculus, in courses selected from the above areas. Also as a minimum, teachers of mathematics in grades nine through twelve should have completed successfully a program of at least twenty-four semester hours, including a full-year program in calculus, in courses selected from the above areas. Both programs should contain fundamental treatments of relevant topics from the foundations of mathematics and probability and statistics. These programs in mathematics should be supplemented by a basic program in education and psychology. As a minimum, a teacher should have completed successfully eighteen semester hours, including student teaching in mathematics, in such courses as: a methods course in the teaching of mathematics; psychology of learning (with particular reference to adolescents); psychology of adjustment (mental hygiene); and tests and measurements. This total program of specialization should be based on a strong program of general education.

Particularly in this period of great potential change it is imperative that teachers take advantage of the many different opportunities available for in-service training to maintain and increase their proficiency. The National Science Foundation institutes are making a very significant contribution to this program. Many colleges and universities conduct summer sessions and evening schools with programs designed for the training of teachers. Local and regional groups have organized self-study and discussion programs. Organizations such as the National Council of Teachers of Mathematics, the Central Association of Science and Mathematics Teachers, the Mathematical Association of America, and state and regional educational associations offer almost unlimited possibilities for hearing informative addresses and discussions on many different types of curriculum and instructional problems. These same organizations provide abundant opportunity to their membership for keeping currently informed on important developments that are taking place in the interest of improved instruction in mathematics. The periodicals, THE MATHEMATICS TEACHER, *The Arithmetic Teacher*, *School Science and Mathematics*, and *The American Mathematical Monthly*, are the media through which this is accomplished. Administrators should insist that the teachers on their staffs become members of at least one of these organizations, and should give them positive encouragement, including financial aid, to attend and participate in their meetings.

Classroom instruction is a demanding job. To carry out his teaching responsibilities, the classroom teacher needs to have the time and opportunity to know his pupils personally, to give them individual attention, to keep a careful check on their out-of-class work as well as their in-class work, to reflect upon the present day's assignment that he may do the best job of which he is capable in handling it, to consider tomorrow's assignment that it may be presented to the pupils for their greatest benefit, and to read and study for his own enlightenment that he may keep alert and informed for the challenges which inspired instruction can provoke and encourage.

This report has been directed toward the great imbalance in the educational program of the secondary schools of the United States—an imbalance which has been due to a lack of proper emphasis on

the significance of mathematics as an essential part of any carefully constructed educational program. No stronger testimony in support of the proposals of this report can be found than that implied in the provisions of the National Defense Education Act of 1958. This act provides money for the identification and encouragement of able high-school pupils, the strengthening of science, mathematics, and modern foreign language instruction, and assistance to both graduate and undergraduate students.

Sixty million dollars has been authorized for the next four years for testing pupils and guiding the able ones toward college. After the first year of the program the federal funds must be matched by the state on a dollar-for-dollar basis. Also 28 million dollars has been authorized for institutes to improve the qualifications of counselors who guide the able youth toward maximum development.

For the improvement of state supervisory or related services in public elementary and secondary schools in the fields of science, mathematics, and modern foreign languages, twenty million dollars has been authorized. After the first year the state must match dollar for dollar the federal funds, which will total 35 million dollars during the four-year period. Most states already have many special supervisors, but few states have supervisors of mathematics instruction. It is hoped that the money from the National Defense Education Act will permit states to correct this imbalance within their own educational structure.

The provision of such vast sums of money is striking indication of the national concern over the need for a greatly improved program of mathematics in the schools of our nation. This concern and interest demand the thoughtful and determined effort of every individual, committee, and organization whose major concern is the teaching of mathematics. Only through co-operative effort in careful study, searching experimentation, and exacting evaluation can the wise expenditure of such funds be assured. The deliberate intent of this report is to help teachers and administrators in their sincere efforts to comply with the current exhortations of a disturbed society to modify curricula and intensify teaching efforts so that pupils in our nation may benefit from a truly improved program of mathematical education.

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Since the fabric of the world is the most perfect and was established by the wisest Creator, nothing happens in this world in which some reason of maximum or minimum would not come to light.—*Euler*

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The National Council of Teachers of Mathematics as a dynamic, growing organization furnishes leadership in mathematics education in the United States of America and Canada. At present the membership of more than 20,000 comes from all parts of the world. The publications of NCTM are excellent, and give help and inspiration to every member.

In addition to membership in NCTM, mathematics teachers have organized into local groups throughout the country. Each group develops a program of meetings that meets the needs of the members and that is appropriate to the geographic location of the group. Many of the local groups are affiliated with NCTM. This year there are seventy affiliated groups scattered in forty-four states, the District of Columbia, and Ontario, Canada. Three new groups have affiliated with NCTM within the past year. They are: Pinellas County (Florida) Progressive Council of Mathematics Teachers, Alamo District (Texas) Council of Teachers of Mathematics, Florida State University Mathematics Teaching Club. The South Carolina Council of Mathematics Teachers has been reinstated as an affiliated group.

The officers of an affiliated group are very important to the group and to NCTM. The success of the local group depends upon their plans and work. The National Council depends upon the officers of the affiliated group for communication with the group's members.

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The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of The Mathematics Teacher. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

JOINT MEETING WITH NEA

July 1, 1959
St. Louis, Missouri
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NINETEENTH SUMMER MEETING

August 17–19, 1959
University of Michigan, Ann Arbor, Michigan
Phillip S. Jones, Mathematics Department, University of Michigan, Ann Arbor, Michigan

THIRTY-EIGHTH ANNUAL MEETING

April 20–23, 1960
Hotel Statler, Buffalo, New York
Louis F. Scholl, Board of Education, Buffalo 2, New York

Other professional dates

Women's Mathematics Club of Chicago and Vicinity

May 16, 1959
Stouffer's Restaurant, Randolph and Michigan, Chicago, Illinois
Delphine Lipecki, Glenbrook High School, Northbrook, Illinois

Twelfth Annual Workshop for Mathematics Teachers

June 15–27, 1959
Indiana University, Bloomington, Indiana
Philip Peak, School of Education, Indiana University, Bloomington, Indiana

Tenth Annual Institute for Teachers of Mathematics, sponsored by ATMNE

August 12–19, 1959
University of Rhode Island, Kingston, Rhode Island
Margaret C. Conneely, One Mayfair Drive, Rumford 16, Rhode Island

Fortieth Summer Meeting, Mathematical Association of America

August 31–September 3, 1959
University of Utah, Salt Lake City, Utah
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